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In ΔABC the following relationship holds:

$$\sqrt[3]{\prod_{cyc}\left(1+\frac{1}{m_a}+\frac{1}{m_b}\right)} \geq \frac{\sum(a+2)sin\frac{A}{2}}{\sum asin\frac{A}{2}} > 1+\frac{2}{s}$$

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Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco, Solution 2 by proposer

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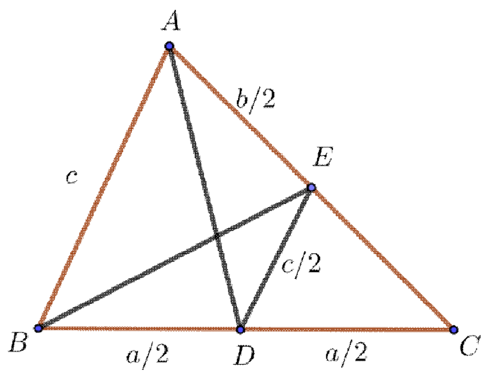

Ptolemy's inequality in $ABDE$: $c\cdot\frac{c}{2}+\frac{a}{2}\cdot\frac{b}{2}\geq m_a m_b \Rightarrow$

$$m_a m_b \overset{(1)}{\leq} \frac{1}{4}(2c^2+ab)(and\ analogs)$$

$$\frac{\sum(a+2)\sin\frac{A}{2}}{\sum a\sin\frac{A}{2}} = 1+\frac{2\sum\sin\frac{A}{2}}{\sum a\sin\frac{A}{2}}$$

$a\geq b\geq c\Rightarrow \sin\frac{A}{2}\geq\sin\frac{B}{2}\geq\sin\frac{C}{2}$. By Chebyshev's inequality:

$$\sum_{cyc} a\cdot\sin\frac{A}{2}\geq\frac{1}{3}\left(\sum_{cyc} a\right)\left(\sum_{cyc}\sin\frac{A}{2}\right)\Rightarrow\frac{\sum(a+2)\sin\frac{A}{2}}{\sum a\sin\frac{A}{2}}\leq 1+\frac{3}{s}$$

$$\sqrt[3]{\prod_{cyc}\left(1+\frac{1}{m_a}+\frac{1}{m_b}\right)}\overset{AM-GM}{\geq}\sqrt[3]{\prod_{cyc}\left(1+\frac{2}{\sqrt{m_a m_b}}\right)}\overset{(1)}{\geq}\sqrt[3]{\prod_{cyc}\left(1+\frac{4}{\sqrt{2c^2+ab}}\right.}\overset{Holder}{\geq}$$

$$\geq 1+\sqrt[3]{\prod_{cyc}\frac{4}{\sqrt{2c^2+ab}}}=1+\frac{4}{\sqrt[6]{\prod_{cyc}(2c^2+ab)}}\overset{(*)}{\geq}1+\frac{3}{s}$$

$$(*)\Leftrightarrow 16s^2\geq 9\sqrt[3]{\prod_{cyc}(2c^2+ab)}$$

$$3\sqrt[3]{\prod_{cyc}(2c^2+ab)}\overset{AM-GM}{\leq}\sum_{cyc}(2c^2+ab)=2\sum_{cyc}a^2+\sum_{cyc}ab$$

$$\Rightarrow 9\sqrt[3]{\prod_{cyc}(2c^2+ab)}\leq 6\sum_{cyc}a^2+3\sum_{cyc}ab=15s^2-9r^2-36Rr\Rightarrow$$

$$16s^2\geq 9\sqrt[3]{\prod_{cyc}(2c^2+ab)}$$

$$\Rightarrow\sqrt[3]{\prod_{cyc}\left(1+\frac{1}{m_a}+\frac{1}{m_b}\right)}\geq\frac{\sum(a+2)\sin\frac{A}{2}}{\sum a\sin\frac{A}{2}}$$

$$\frac{\sum(a+2)\sin\frac{A}{2}}{\sum a\sin\frac{A}{2}}=1+\frac{2\sum\sin\frac{A}{2}}{\sum a\cdot\sin\frac{A}{2}}\overset{a<s}{>}1+\frac{2\sum\sin\frac{A}{2}}{s\cdot\sum\sin\frac{A}{2}}\Rightarrow$$

$$\frac{\sum(a+2)\sin\frac{A}{2}}{\sum a\sin\frac{A}{2}} > 1+\frac{2}{s}$$

Therefore,

$$\sqrt[3]{\prod_{cyc}\left(1+\frac{1}{m_a}+\frac{1}{m_b}\right)} \ge \frac{\sum(a+2)\sin\frac{A}{2}}{\sum a\sin\frac{A}{2}} > 1+\frac{2}{s}$$

Solution 2 by proposer

$$\because \sum_{i=1}^{2}(a_i+b_i)\sum_{i=1}^{2}\frac{a_ib_i}{a_i+b_i} \le \left(\sum_{i=1}^{2}a_i\right)\left(\sum_{i=1}^{2}b_i\right) \ (Milne's\ ineq.\ n=2)$$

$$\sum_{cyc}(m_a+m_b)\sum_{cyc}\left(\frac{m_am_b}{m_a+m_b}\right) \le \left(\sum_{cyc}m_a\right)\left(\sum_{cyc}m_b\right) \Rightarrow$$

$$\sum_{cyc}\left(\frac{m_am_b}{m_a+m_b}\right) \ge \frac{(\sum m_a)^2}{2\sum m_a} \Rightarrow \sum_{cyc}\left(\frac{m_am_b}{m_a+m_b}\right) \le \frac{1}{2}\sum_{cyc}m_a \le \frac{1}{2}\sum_{cyc}\frac{b+c}{2} = \frac{2s}{2} = s$$

$$\because m_a \le \frac{b+c}{2}, m_b \le \frac{c+a}{2}, m_c \le \frac{a+b}{2}$$

$$\sum_{cyc}\frac{1}{x+\frac{m_am_b}{m_a+m_b}} \overset{BCS}{\ge} \frac{9}{3x+\sum\frac{m_am_b}{m_a+m_b}} \ge \frac{9}{3x+s} = \frac{3}{x+\frac{s}{3}} \Rightarrow$$

$$\int_0^1\sum_{cyc}\frac{1}{x+\frac{m_am_b}{m_a+m_b}}dx \ge \int_0^1\frac{3}{x+\frac{s}{3}}dx \Rightarrow$$

$$\sum_{cyc}\log\left(1+\frac{m_a+m_b}{m_am_b}\right) \ge 3\log\left(1+\frac{3}{s}\right) \Rightarrow \sqrt[3]{\prod_{cyc}\left(1+\frac{1}{m_a}+\frac{1}{m_b}\right)} \ge 1+\frac{3}{s};(1)$$

Now,

$$(a+b+c)\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right) \overset{Chebyshev's}{\le} 3a\sin\frac{A}{2}+3b\sin\frac{B}{2}+3c\sin\frac{C}{2}$$

$$\Leftrightarrow \sum_{cyc}(a-b)\left(\sin\frac{A}{2}-\sin\frac{B}{2}\right) \ge 0, (2)$$

On the other hand,

$$\sum_{cyc}(a+b-c)\sin\frac{C}{2}>0,(3)$$

From (2),(3) it follows that:

$$\frac{1}{2}\cdot\frac{3}{s}\geq\frac{\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}}{a\sin\frac{A}{2}+b\sin\frac{B}{2}+c\sin\frac{C}{2}}>\frac{1}{s};(4)$$

From (1),(4) it follows that:

$$\sqrt[3]{\prod_{cyc}\left(1+\frac{1}{m_a}+\frac{1}{m_b}\right)}\geq 1+\frac{3}{s}\geq 1+\frac{2\sum\sin\frac{A}{2}}{\sum a\sin\frac{A}{2}}=\frac{\sum(a+2)\sin\frac{A}{2}}{\sum a\sin\frac{A}{2}}>1+\frac{2}{s}$$