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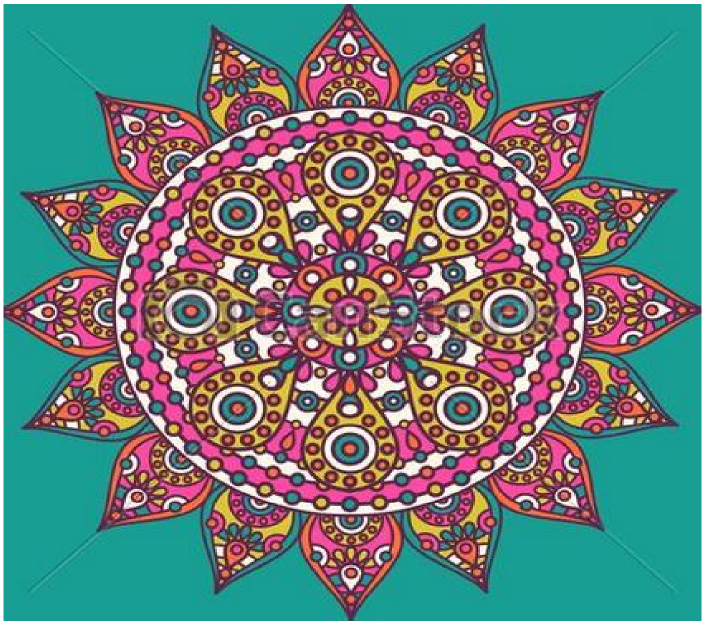

In ΔABC, n_a –Nagel's cevian, the following relationship holds:

$$\frac{h_a}{n_a}+\frac{h_b}{n_b}+\frac{h_c}{n_c}\geq\frac{9r}{\sqrt{s^2-2r(4R+r)}}$$

Proposed by Bogdan Fuştei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$Stewart's\ theorem\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$

$$=an_a^2+a(as-s^2)$$

$$\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+s(2bc\cos A-2bc)$$

$$=as^2-4sbc\sin^2\frac{A}{2}=as^2-4s(s-b)(s-c)$$

$$and\ analogs\Rightarrow\sum an_a^2=s^2(2s)-4s\sum(s-b)(s-c)=s^2(2s)-4s(4Rr+r^2)$$

$$\Rightarrow\sum an_a^2=2s(s^2-8Rr-2r^2)$$

$$and \sqrt[3]{\prod an_a^2} \overset{GM \le AM}{\le} \frac{1}{3}\sum an_a^2 = \frac{2s(s^2-8Rr-2r^2)}{3}$$

$$\Rightarrow \prod an_a^2 \overset{(i)}{\le} \frac{8s^3(s^2-8Rr-2r^2)^3}{27}$$

$$Also, 2s^2 \overset{Gerretsen}{\ge} 27Rr + 5r(R-2r) \overset{Euler}{\ge} 27Rr \Rightarrow 2s^2 \overset{(ii)}{\ge} 27Rr$$

$$Now, \frac{h_a}{n_a} + \frac{h_b}{n_b} + \frac{h_c}{n_c} \overset{AM \ge GM}{\ge} 3\sqrt[3]{\frac{h_a h_b h_c}{n_a n_b n_c}} = \frac{6rs}{\sqrt[3]{abcn_a n_b n_c}} \overset{?}{\ge} \frac{9r}{\sqrt{s^2-2r(4R+r)}}$$

$$\Leftrightarrow 64s^6(s^2-8Rr-2r^2)^3 \underset{(iii)}{\overset{?}{\ge}} 729abc\left(\prod an_a^2\right)$$

$$Now, RHS\ of\ (iii) \overset{via\ (i)}{\le} 729.4Rrs.\frac{8s^3(s^2-8Rr-2r^2)^3}{27} \overset{?}{\le} 64s^6(s^2-8Rr-2r^2)^3$$

$$\Leftrightarrow 2s^2 \overset{?}{\ge} 27Rr \to true\ via\ (ii) \Rightarrow (iii)\ is\ true$$

$$\therefore \frac{h_a}{n_a} + \frac{h_b}{n_b} + \frac{h_c}{n_c} \ge \frac{9r}{\sqrt{s^2-2r(4R+r)}} (QED)$$