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In ΔABC the following relationship holds:

$$1^o.\quad \sqrt{\frac{m_b m_c}{r_a}}+\sqrt{\frac{m_c m_a}{r_b}}+\sqrt{\frac{m_a m_b}{r_c}}\le\frac{4R+r}{\sqrt{3r}}$$

$$2^o.\quad m_a^2+m_b^2+m_c^2\ge w_a^2+w_b^2+w_c^2+\frac{r}{4R}\cdot\frac{h_a h_b h_c}{m_a m_b m_c}\cdot(R^2-4r^2)$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$1^o.\quad \sqrt{\frac{m_b m_c}{r_a}}+\sqrt{\frac{m_c m_a}{r_b}}+\sqrt{\frac{m_a m_b}{r_c}}\overset{BCS}{\le}\sqrt{\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}\frac{1}{r_a}\right)}=\sqrt{\frac{1}{r}\sum_{cyc}m_b m_c}\le$$

$$\overset{\sum xy\le\frac{1}{3}(\sum x)^2}{\le}\sqrt{\frac{1}{3r}\left(\sum_{cyc}m_a\right)^2}=\frac{1}{\sqrt{3r}}\sum_{cyc}m_a$$

$$2^o.\quad \frac{r}{4R}\cdot\frac{h_a h_b h_c}{m_a m_b m_c}=\frac{r}{4R\cdot m_a m_b m_c}\cdot\frac{8s^3r^3}{4Rrs}=\frac{s^2r^3}{2R^2\cdot m_a m_b m_c}\le$$

$$\le\frac{s^2r^3}{2R^2\cdot s^2 r}\Rightarrow\frac{r}{4R}\cdot\frac{h_a h_b h_c}{m_a m_b m_c}\le r^2/2R^2$$

$$\because \sum_{cyc} w_a^2 \le \sum_{cyc} s(s-a) = s^2$$

$$\because \sum_{cyc} m_a^2 = \frac{3}{4}\sum_{cyc} a^2 = \frac{3}{2}(s^2 - r^2 - 4Rr)$$

It is suffices to prove that: $\frac{3}{2}(s^2 - r^2 - 4Rr) \ge s^2 + \frac{r^2}{2R^2}(R^2 - 4r^2) \Leftrightarrow$

$$s^2R^2 \ge 12R^3r + 4R^2r^2 - 4r^4$$

By GERRETSEN: $s^2R^2 \ge (16R - 5r)rR^2 \overset{(*)}{\ge} 12R^3r + 4R^2r^2 - 4r^4$

$$(*) \Leftrightarrow 4R^3 - 9R^2r + 4r^3 \ge 0 \Leftrightarrow (R - 2r)(4R^2 - Rr - 2r^2) \ge 0$$

$\Leftrightarrow (R - 2r)\left[3R^2 + \frac{1}{2}R(R - 2r) + \frac{1}{2}(R - 2r)(R + 2r)\right] \ge 0$, which is true from

$R \ge 2r\ (EULER)$.

Therefore,

$$m_a^2 + m_b^2 + m_c^2 \ge w_a^2 + w_b^2 + w_c^2 + \frac{r}{4R} \cdot \frac{h_a h_b h_c}{m_a m_b m_c} \cdot (R^2 - 4r^2)$$

Note by editor:

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