ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

In any ΔABC the following relationship holds:

$$27r^3\sqrt{\frac{2r}{R}} \le \sum (p-b)(p-c)\sqrt{h_b h_c} \le \frac{27R^2 r}{4}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum a(p-b)(p-c) = \sum a(p^2 - p(b+c) + bc)$$

$$= p^2(2p) - 2p(p^2 + 4Rr + r^2) + 12Rrp = 2p(2Rr - r^2)$$

$$\Rightarrow \sum a(p-b)(p-c) \overset{(i)}{=} 2p(2Rr - r^2) \text{ Now, } \sum (p-b)(p-c)\sqrt{h_b h_c}$$

$$= \sum (p-b)(p-c)\sqrt{\frac{a.4Rrp}{4R^2}}$$

$$= \sqrt{\frac{rp}{R}}\sum\left(\sqrt{(p-b)(p-c)}\sqrt{a(p-b)(p-c)}\right) \overset{CBS}{\le} \sqrt{\frac{rp}{R}}\sqrt{\sum (p-b)(p-c)}\sqrt{\sum a(p-b)(p-c)} \overset{\text{uisng (i)}}{\le} \sqrt{\frac{rp}{R}}\sqrt{4Rr + r^2}\sqrt{2p(2Rr - r^2)}$$

$$\overset{\text{Mitrinovic}}{\le} r\sqrt{\frac{27R^2(4R+r)(2Rr-r^2)}{2R}} \overset{?}{\le} \frac{27R^2 r}{4} \Leftrightarrow \frac{27R^2(4R+r)(2Rr-r^2)}{2R} \overset{?}{\le} \frac{729R^4}{16}$$

$$\Leftrightarrow 27R^3 \overset{?}{\ge} 8(4R+r)(2Rr-r^2)$$

$$\Leftrightarrow 27t^3 - 64t^2 + 16t + 8 \overset{?}{\ge} 0 \Leftrightarrow (t-2)\left((t-2)(27t+8) + 84\right) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\therefore \sum (p-b)(p-c)\sqrt{h_b h_c} \le \frac{27R^2 r}{4}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

$$\sum(p-b)(p-c)\sqrt{h_bh_c}\overset{A-G}{\geq}3\sqrt[3]{\left((p-a)(p-b)(p-c)\right)^2\left(\frac{8r^3p^3}{4Rrp}\right)}\overset{Mitrinovic}{\geq}3\sqrt[3]{r^4\left(\frac{2r^2}{R}\right)(27r^2)^2}=27r^3\sqrt{\frac{2r}{R}}$$

$$\therefore 27r^3\sqrt{\frac{2r}{R}}\leq\sum(p-b)(p-c)\sqrt{h_bh_c}\ \text{(Proved)}$$