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In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{h_a}{bc}cos^2\frac{A}{2}\le\sum_{cyc}\frac{r_a}{bc}cos^2\frac{A}{2}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Marian Ursărescu-Romania, Solution 2 by Alex Szörös-Romania

Solution 1 by Marian Ursărescu-Romania

$$\sum_{cyc}\frac{h_a}{bc}cos^2\frac{A}{2}=\sum_{cyc}\frac{2F}{abc}cos^2\frac{A}{2}=\sum_{cyc}\frac{2F}{4Rrs}cos^2\frac{A}{2}=\frac{1}{2R}\sum_{cyc}cos^2\frac{A}{2}=$$

$$=\frac{1}{2Rr}\cdot\frac{4R+r}{2R}=\frac{4R+r}{4R^2}$$

We must show that:

$$\sum_{cyc}\frac{r_a}{bc}cos^2\frac{A}{2}\ge\frac{4R+r}{4R^2};(1)$$

$$\because\sum_{cyc}\frac{r_a}{bc}cos^2\frac{A}{2}\ge 3\sqrt[3]{\prod_{cyc}\frac{r_a}{bc}cos^2\frac{A}{2}};(2)$$

$$r_a r_b r_c = s^2 r, abc = 4Rrs \text{ and } \cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2} = \frac{s^2}{16R^2}; (3)$$

From (2),(3) we must to prove that:

$$\sum_{cyc}\frac{r_a}{bc}\cos^2\frac{A}{2} \ge 3\sqrt[3]{\frac{s^2 r\cdot s^2}{16s^2r^2R^2\cdot 16R^2}} \Rightarrow \sum_{cyc}\frac{r_a}{bc}\cos^2\frac{A}{2} \ge \frac{3}{4}\sqrt[3]{\frac{s^2}{4R^4 r}}; (4)$$

From Coşniţă-Turtoiu: $s^2 \ge \frac{27}{2}Rr; (5)$

From (4),(5) it follows that:

$$\sum_{cyc}\frac{r_a}{bc}\cos^2\frac{A}{2} \ge \frac{9}{8}\cdot\frac{1}{R}; (6)$$

From (1),(6) we must to prove that:

$$\frac{9}{8}\cdot\frac{1}{R} \ge \frac{4R+r}{4R^2} \Leftrightarrow R \ge 2r(Euler).$$

Solution 2 by Alex Szörös-Romania

$$ah_a = 2F = bc\sin A \Rightarrow \frac{h_a}{bc} = \frac{\sin A}{a} = \frac{1}{2R}$$

$$\sum_{cyc}\frac{h_a}{bc}\cos^2\frac{A}{2} = \frac{1}{2R}\sum_{cyc}\frac{1+\cos A}{2} = \frac{1}{4R}\left(3+\sum_{cyc}\cos A\right) = \frac{1}{4R}\cdot\left(4+\frac{r}{R}\right) = \frac{4R+r}{4R^2}; (1)$$

$$\sum_{cyc}\frac{r_a}{bc}\cos^2\frac{A}{2} = \sum_{cyc}\frac{F}{s-a}\cdot\frac{1}{bc}\cdot\frac{s(s-a)}{bc} = sF\sum_{cyc}\frac{1}{b^2c^2} = \frac{sF}{(abc)^2}\sum_{cyc}a^2 =$$

$$= \frac{sF}{(4RF)^2}\sum_{cyc}a^2 = \frac{s(2s^2-2r^2-8Rr)}{8R^2 r} = \frac{s^2-r^2-4Rr}{8R^2 r} \ge \frac{16Rr-5r^2-r^2-4Rr}{8R^2 r} =$$

$$= \frac{12Rr-6r^2}{8R^2 r} \Rightarrow \sum_{cyc}\frac{r_a}{bc}\cos^2\frac{A}{2} \ge \frac{3(2R-r)}{4R^2}; (2)$$

From (1),(2) we must to prove that:

$$\frac{3(2R-r)}{4R^2} \ge \frac{4R+r}{4R^2} \Leftrightarrow 5R-3r \ge 4R+r \Leftrightarrow R \ge 2r(Euler); (3)$$

From (3), it follows that:

$$\sum_{cyc}\frac{r_a}{bc}\cos^2\frac{A}{2} \ge \frac{3(2R-r)}{4R^2} \ge \frac{4R+r}{4R^2} = \sum_{cyc}\frac{h_a}{bc}\cos^2\frac{A}{2}$$

Therefore,

$$\sum_{cyc}\frac{h_a}{bc}cos^2\frac{A}{2}\le\sum_{cyc}\frac{r_a}{bc}cos^2\frac{A}{2}$$

Note by editor:

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