R M M

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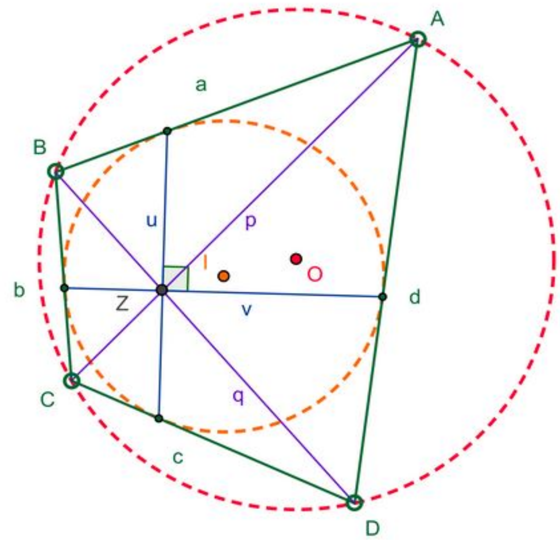

a, b, c, d –sides, r –inradii, R –circumradii, F –area in a bicentric quadrilateral. Prove that:

$$a^4 + b^4 + c^4 + d^4 \geq 8F^2\left(1 - \sqrt{\frac{r}{R}}\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We know that: $a + c = b + d; r = \frac{\sqrt{abcd}}{a+c}; F = \sqrt{abcd} = sr$

$s = \frac{\sum a}{2} = b + d \overset{(1)}{\geq} 4r = 4\frac{\sqrt{abcd}}{a+c}; (1) \Leftrightarrow (a + c)(b + d) \geq 4\sqrt{abcd}$, which is true from

AM-GM: $a + c \geq 2\sqrt{ac}, b + d \geq 2\sqrt{bd} \Rightarrow s \geq 4r; (I)$

$$\sum_{cyc} a^4 \overset{Holder}{\geq} \frac{(\sum a)^4}{4^3} = \frac{s^4}{4} \overset{(2)}{\geq} 8F^2\left(1 - \sqrt{\frac{r}{R}}\right); \ (2) \Leftrightarrow s^2 \geq 32r^2\left(1 - \sqrt{\frac{r}{s}}\right) \Leftrightarrow$$

$$s^2 + 32r^2\sqrt{\frac{r}{s}} \geq 32r^2$$

By AM-GM: $s^2 + 32r^2\sqrt{\frac{r}{s}} \geq 2\sqrt{s^2 \cdot 32r^2\sqrt{\frac{r}{s}}} = 8r\sqrt{2s\sqrt{sr}} \overset{(I)}{\geq} 8r\sqrt{2 \cdot 4r\sqrt{4r^2}} = 32r^2$

Therefore,

$$a^4+b^4+c^4+d^4 \geq 8F^2\left(1-\sqrt{\frac{r}{R}}\right)$$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solution.