R M M

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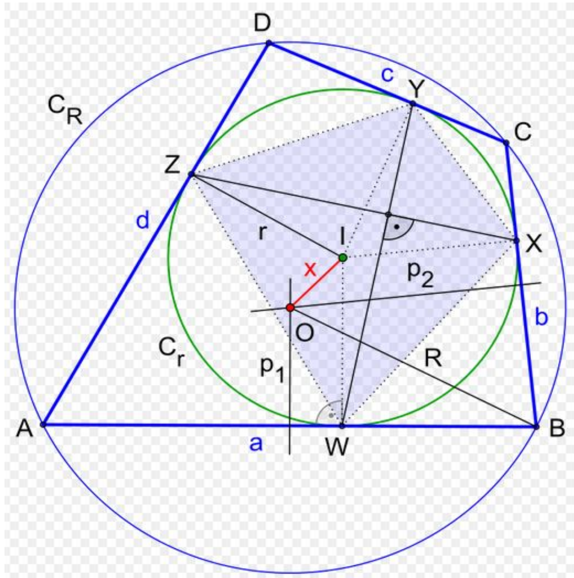

a, b, c, d – sides, r – inradii in a bicentric quadrilateral. Prove that:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{d}+\frac{d^2}{a}\geq 8r$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Alex Szoros-Romania, Solution 2 by Adrian Popa-Romania

Solution 1 by Alex Szoros-Romania

If we denote F – area of this quadrilateral, then: $4r^2\leq F=\sqrt{abcd}$

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{d}+\frac{d^2}{a}\overset{Bergstrom}{\geq}\frac{(a+b+c+d)^2}{a+b+c+d}=a+b+c+d\geq$$

$$4\sqrt[4]{abcd}=4\sqrt{F}\geq 4\sqrt{4r^2}=8r$$

Observation:

$ABCD$ – inscriptible quadrilateral, then $F=\sqrt{(s-a)(s-b)(s-c)(s-d)}$

$$2s=a+b+c+d=2(a+c)=2(b+d)\Rightarrow$$

$$F=\sqrt{(a+c-a)(b+d-b)(a+c-c)(b+d-d)}=\sqrt{abcd}$$

$$\because r=\frac{F}{s}\Leftrightarrow r=\frac{2\sqrt{abcd}}{a+b+c+d}$$

But: $(a+b+c+d)^2=[(a+b)+(c+d)]^2\geq 4(a+b)(c+d)\geq$

$$\geq 4\cdot 2\sqrt{ab}\cdot 2\sqrt{cd} = 16\sqrt{abcd}$$

$$(2s)^2 \geq 16F \Rightarrow s^2 \geq 4F \Rightarrow \left(\frac{F}{r}\right)^2 \geq 4F \Rightarrow F \geq 4r^2$$

Therefore,

$$F \geq \sqrt{abcd} \geq 4r^2$$

Solution 2 by Adrian Popa-Romania

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{d}+\frac{d^2}{a} \overset{Bergstrom}{\geq} \frac{(a+b+c+d)^2}{a+b+c+d} = a+b+c+d \geq$$

$$r = \frac{\sqrt{abcd}}{a+c} = \frac{\sqrt{abcd}}{\frac{a+b+c+d}{2}} = \frac{2\sqrt{abcd}}{a+b+c+d}$$

We must to prove that:

$$a+b+c+d \geq \frac{16\sqrt{abcd}}{a+b+c+d} \Leftrightarrow (a+b+c+d)^2 \geq 16\sqrt{abcd}$$

which is true from AM-GM: $\frac{a+b+c+d}{4} \overset{AM-GM}{\geq} \sqrt[4]{abcd}$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solution.