INEQUALITIES WITH CONVEX FUNCTIONS

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ABSTRACT. In this paper it is proved a property of convex functions which can generate beautiful inequalities.

Main result:
Theorem 1
If $a, b, x, y, z \in \mathbb{R}; a \leq x \leq y \leq z \leq b; y + z \leq b + x; f : [a, b] \to \mathbb{R}; f$ convexe, then:

$$f(y) + f(z) \leq f(x) + f(y + z - x) \tag{1}$$

Proof.

$$x \leq z \Rightarrow 0 \leq z - x \Rightarrow y \leq y + z - x$$
$$x \leq y \Rightarrow 0 \leq y - x \Rightarrow z \leq y + z - x$$
$$y \in [x; y + z - x] \Rightarrow (\exists)\alpha \in [0, 1]$$

$$y = \alpha x + (1 - \alpha)(y + z - x) \tag{2}$$

$$z \in \beta x + (1 - \beta)(y + z - x) \tag{3}$$

By adding $(2); (3)$:

$$y + z = (\alpha + \beta)x + (y + z - x)(2 - \alpha - \beta) \tag{4}$$

$$y + z = x + (y + z - x)$$

Replace in (4):

$$x + (y + z - x) = (\alpha + \beta)x + (y + z - x)(2 - \alpha - \beta)$$
$$(\alpha + \beta - 1)x + (y + z - x)(1 - \alpha - \beta) = 0$$
$$(\alpha + \beta - 1)(x - y - z + x) = 0$$
$$(\alpha + \beta - 1)(2x - y - z) = 0$$

$$\text{Case I: } 2x - y - z = 0 \Rightarrow y + z = 2x \tag{5}$$

$$\text{But } x \leq y; x \leq z \Rightarrow 2x \leq y + z \tag{6}$$

By $(5), (6) \Rightarrow x = y = z$.
Inequality (1) becomes:

$$f(x) + f(x) \leq f(x) + f(x + x - x) \text{ (true)}$$
$$\text{Case II: } \alpha + \beta - 1 = 0 \Rightarrow \alpha + \beta = 1 \Rightarrow 2 - \alpha - \beta = 1$$
$$f \text{ convexe; } y, z \in [x, y + z - x] \Rightarrow$$
$$\Rightarrow (\exists)\alpha, \beta \in [0, 1] \text{ such that:}$$

$$f(y) \leq \alpha f(x) + (1 - \alpha)f(y + z - x) \tag{7}$$

$$f(z) \leq \beta f(x) + (1 - \beta)f(y + z - x) \tag{8}$$

By adding (7); (8):

$$f(y)+f(z)\leq(\alpha+\beta)f(x)+(2-\alpha-\beta)f(y+z-x)$$

$$f(y)+f(z)\leq f(x)+f(y+z-x)$$

□

Theorem 2

If $a,b,x,y,z\in\mathbb{R}; a\leq x\leq y\leq z\leq b; y+z\leq b+x; f:[a,b]\to\mathbb{R}; f$ concave, then:

(9) $$f(y)+f(z)\geq f(x)+f(y+z-x)$$

Application 1:

If $a,b,x,y,z\in\mathbb{R}; 0<a\leq x\leq y\leq z\leq b; y+z\leq b+x; n\in\mathbb{N}\setminus\{0\}$ then:

(10) $$\frac{1}{y^n}+\frac{1}{z^n}\leq\frac{1}{x^n}+\frac{1}{(y+z-x)^n}$$

Proof.

$$\text{We take in (1)}: f(x)=\frac{1}{x^n}; f'(x)=\frac{-nx^{n-1}}{x^{2n}}$$

$$f'(x)=\frac{-n}{x^{n+1}}; f''(x)=\frac{n(n+1)x^n}{x^{2n+2}}=\frac{n(n+1)}{x^{n+1}}>0$$

f convexe

$$f(y)+f(z)\leq f(x)+f(y+z-x)$$

$$\frac{1}{y^n}+\frac{1}{z^n}\leq\frac{1}{x^n}+\frac{1}{(y+z-x)^n}$$

Equality holds for $x=y=z$. □

Application 2:

If $a,b,x,y,z\in\mathbb{R}; 0<a\leq x\leq y\leq z\leq b; y+z\leq b+x; n\in\mathbb{N}\setminus\{0\}$ then:

$$\sqrt[n]{y}+\sqrt[n]{z}\leq\sqrt[n]{x}+\sqrt[n]{y+z-x}$$

Proof.

$$\text{We take in (9)}: f(x)=\sqrt[n]{x}; f'(x)=\frac{1}{n}x^{\frac{1}{n}-1}$$

$$f''(x)=\frac{1}{n}\Big(\frac{1}{n}-1\Big)x^{\frac{1}{n}-2}<0; f\text{ concave}$$

$$f(y)+f(z)\geq f(x)+f(y+z-x); f:(0,\infty)\to\mathbb{R}$$

(11) $$\sqrt[n]{y}+\sqrt[n]{z}\geq\sqrt[n]{x}+\sqrt[n]{y+z-x}$$

Equality holds for $x=y=z$. □

Application 3:

If $a,b,x,y,z\in\mathbb{R}; 0<a\leq x\leq y\leq z\leq b; y+z\leq b+x; n\in\mathbb{N}\setminus\{0\}$ then:

(12) $$\operatorname{arccot}y+\operatorname{arccot}z\leq\operatorname{arccot}x+\operatorname{arccot}(y+z-x)$$

Proof. We take in (1) : $f(x) = \operatorname{arccot} x$;

$$f'(x) = \frac{-1}{1+x^2}; f''(x) = \frac{2x}{(1+x^2)} > 0; f \text{ convexe}$$

$$f(y) + f(z) \leq f(x) + f(y+z-x); f : (0, \infty) \to \mathbb{R}$$

$$\operatorname{arccot} y + \operatorname{arccot} z \leq \operatorname{arccot} x + \operatorname{arccot}(y+z-x)$$

Equality holds for: $x = y = z$. □

Application 4:

If $a, b, x, y, z \in \mathbb{R}; a \leq x \leq y \leq z \leq b; y + z \leq b + x; n \in \mathbb{N} \setminus \{0\}$ then:

$$e^{y^2} + e^{z^2} \leq e^{x^2} + e^{(y+z-x)^2} \tag{13}$$

Proof. We take in (1) : $f(x) = e^{x^2}$; $f'(x) = 2xe^{x^2}$

$$f''(x) = 2e^{x^2} + 4x^2e^{x^2} = 2e^{x^2}(1 + 2x^2) > 0; f \text{ convexe}$$

$$f(y) + f(z) \leq f(x) + f(y+z-x)$$

$$e^{y^2} + e^{z^2} \leq e^{x^2} + e^{(y+z-x)^2}$$

Equality holds for: $x = y = z$. □

Application 5:
If $0 < a \leq b; n \in \mathbb{N} \setminus \{0\}$ then:

$$\frac{1}{(\sqrt{ab})^n} + \frac{1}{(\frac{a+b}{2})^n} \leq \frac{1}{(\frac{2ab}{a+b})^n} + \frac{1}{(\sqrt{ab} + \frac{a+b}{2} - \frac{2ab}{a+b})^n}$$

Proof. We take in (10):

$$x = \frac{2ab}{a+b}; y = \sqrt{ab}; z = \frac{a+b}{2}$$

By AM-GM-HM:

$$0 < a \leq \frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq b$$

$$\frac{1}{y^n} + \frac{1}{z^n} \leq \frac{1}{x^n} + \frac{1}{(y+z-x)^n}$$

$$\frac{1}{(\sqrt{ab})^n} + \frac{1}{(\frac{a+b}{2})^n} \leq \frac{1}{(\frac{2ab}{a+b})^n} + \frac{1}{(\sqrt{ab} + \frac{a+b}{2} - \frac{2ab}{a+b})^n}$$

Equality holds for $a = b$. □

Application 6:
If $0 < a \leq b; n \in \mathbb{N} \setminus \{0\}$ then:

$$\sqrt[2n]{ab} + \sqrt[n]{\frac{a+b}{2}} \geq \sqrt[n]{\frac{2ab}{a+b}} + \sqrt[n]{\sqrt{ab} + \frac{a+b}{2} - \frac{2ab}{a+b}}$$

Proof. We take in (11):

$$x = \frac{2ab}{a+b}; y = \sqrt{ab}; z = \frac{a+b}{2}$$

By AM-GM-HM:

$$0 < a \leq \frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq b$$

$$\sqrt[n]{y} + \sqrt[n]{z} \geq \sqrt[n]{x} + \sqrt[n]{y+z-x}$$

$$\sqrt[n]{\sqrt{ab}} + \sqrt[n]{\frac{a+b}{2}} \geq \sqrt[n]{\frac{2ab}{a+b}} + \sqrt[n]{\sqrt{ab} + \frac{a+b}{2} - \frac{2ab}{a+b}}$$

$$\sqrt[n]{ab} + \sqrt[n]{\frac{a+b}{2}} \geq \sqrt[n]{\frac{2ab}{a+b}} + \sqrt[n]{\sqrt{ab} + \frac{a+b}{2} - \frac{2ab}{a+b}}$$

Equality holds for $a = b$. □

Application 7:
If $0 < a \leq b$ then:

$$\operatorname{arccot}(\sqrt{ab}) + \operatorname{arccot}\left(\frac{a+b}{2}\right) \leq \operatorname{arccot}\left(\frac{2ab}{a+b}\right) + \operatorname{arccot}\left(\sqrt{ab} + \frac{a+b}{2} - \frac{2ab}{a+b}\right)$$

Proof. We take in (12):

$$x = \sqrt{ab}; y = \frac{a+b}{2}; z = \frac{2ab}{a+b}$$

By AM-GM-HM:

$$0 < a \leq \frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq b$$

$$\operatorname{arccot} y + \operatorname{arccot} z \leq \operatorname{arccot} x + \operatorname{arccot}(y+z-x)$$

$$\operatorname{arccot}(\sqrt{ab}) + \operatorname{arccot}\left(\frac{a+b}{2}\right) \leq \operatorname{arccot}\left(\frac{2ab}{a+b}\right) + \operatorname{arccot}\left(\sqrt{ab} + \frac{a+b}{2} - \frac{2ab}{a+b}\right)$$

Equality holds for $a = b$. □

Application 8:
If $0 < a \leq b$ then:

$$e^{ab} + e^{(\frac{a+b}{2})^2} \leq e^{(\frac{2ab}{a+b})^2} + e^{(\sqrt{ab} + \frac{a+b}{2} - \frac{2ab}{a+b})^2}$$

Proof. We take in (13):

$$x = \frac{2ab}{a+b}; y = \sqrt{ab}; z = \frac{a+b}{2}$$

$$e^{y^2} + e^{z^2} \leq e^{x^2} + e^{(y+z-x)^2}$$

$$e^{(\sqrt{ab})^2} + e^{(\frac{a+b}{2})^2} \leq e^{(\frac{2ab}{a+b})^2} + e^{(\sqrt{ab} + \frac{a+b}{2} - \frac{2ab}{a+b})^2}$$

$$e^{ab} + e^{(\frac{a+b}{2})^2} \leq e^{(\frac{2ab}{a+b})^2} + e^{(\sqrt{ab} + \frac{a+b}{2} - \frac{2ab}{a+b})^2}$$

Equality holds for $a = b$. □

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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