INEQUALITIES WITH CIRCUMPEDAL EXTENSIONS OF THREE CONCURRENT CEVIANS IN A TRIANGLE

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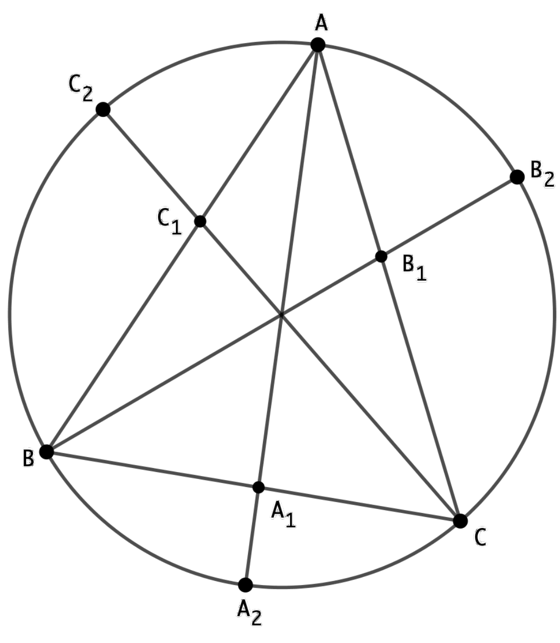
Abstract. In this paper we define the circumpedal extensions for concurrent cevians in a triangle and give some applications of its.

Main result:
Let k_a, k_b, k_c be the circumpedal extensions of three concurrent cevians in ΔABC. Denote $AA_1 = c_a, BB_1 = c_b, CC_1 = c_c, A_1 \in (BC), B_1 \in (CA), C_1 \in (AB)$ and A_2, B_2, C_2 intersection points between cevians and circumcircle; $(A, A_1, A_2); (B, B_1, B_2); (C, C_1, C_2)$ collinears. If $BA_1 = x; CB_1 = y, AC_1 = z, x \in (0, a], y \in (0, b];$ $z \in (0, c], k_a = AA_2, k_b = BB_2, k_c = CC_2$ then:

$$k_a + k_b + k_c \geq 6\sqrt[3]{xyz(a-x)(b-y)(c-z)} \tag{1}$$

$$k_a \cdot k_b \cdot k_c \geq 8\sqrt{xyz(a-x)(b-y)(c-z)} \tag{2}$$

Proof.



$$\rho(A_1) = A_1B \cdot A_1C = x(a-x) \text{ (power of the point } A_1 \text{ to circumcircle)}$$

$$\rho(A_1) = AA_1 \cdot A_1A_2 = c_a \cdot A_1A_2$$

$$c_a \cdot A_1A_2 = x(a-x) \Rightarrow A_1A_2 = \frac{x(a-x)}{c_a}$$

$$k_a = AA_1 + A_1A_2 = c_a + \frac{x(a-x)}{c_a} \overset{\text{AM-GM}}{\geq} 2\sqrt{c_a \cdot \frac{x(a-x)}{c_a}} = 2\sqrt{x(a-x)}$$

$$k_a \geq 2\sqrt{x(a-x)} \tag{3}$$

$$k_b \geq 2\sqrt{y(a-y)} \tag{4}$$

$$k_c \geq 2\sqrt{z(a-z)} \tag{5}$$

$$k_a \cdot k_b \cdot k_c \overset{(3);(4);(5)}{\geq} 8\sqrt{xyz(a-x)(b-y)(c-z)}$$

$$k_a + k_b + k_c = c_a + \frac{x(a-x)}{c_a} + c_b + \frac{y(b-y)}{c_b} + c_c + \frac{z(c-z)}{c_c} \geq$$

$$\overset{\text{AM-GM}}{\geq} 6 \cdot \sqrt[6]{c_a \cdot c_b \cdot c_c \cdot \frac{x(a-x)}{c_a} \cdot \frac{y(b-y)}{c_b} \cdot \frac{z(c-z)}{c_c}} =$$

$$= 6\sqrt[6]{xyz(a-x)(b-y)(c-z)}$$

□

1. CENTROID

For centroid we denote $AA_2 = g_a; BB_2 = g_b; CC_2 = g_c$

$$\frac{A_1B}{A_1C} = 1 \Rightarrow \frac{A_1B}{A_1C + A_1B} = \frac{1}{2} \Rightarrow A_1B = \frac{a}{2}$$

$$x = \frac{a}{2}; y = \frac{b}{2}; z = \frac{c}{2}$$

$$g_a = m_a + \frac{x(a-x)}{m_a} = m_b + \frac{\frac{a}{2}(a - \frac{a}{2})}{m_a} =$$

$$= m_a + \frac{\frac{a}{2} \cdot \frac{a}{2}}{m_a} = m_a + \frac{a^2}{4m_a}$$

$$g_b = m_b + \frac{b^2}{4m_b}; g_c = m_c + \frac{c^2}{4m_c}$$

The inequalities $(1); (2)$ can be written:

$$g_a g_b g_c \geq 8\sqrt{\frac{a}{2} \cdot \frac{b}{2} \cdot \frac{c}{2}\left(a - \frac{a}{2}\right)\left(b - \frac{b}{2}\right)\left(c - \frac{c}{2}\right)} =$$

$$= 8\sqrt{\frac{abc}{8} \cdot \frac{abc}{8}} = 8\sqrt{\left(\frac{abc}{8}\right)^2} = \frac{8}{8}\sqrt{abc} = abc$$

$$g_a + g_b + g_c \geq 6\sqrt[6]{\frac{a}{2} \cdot \frac{b}{2} \cdot \frac{c}{2}\left(a - \frac{a}{2}\right)\left(b - \frac{b}{2}\right)\left(c - \frac{c}{2}\right)} =$$

$$= 6\sqrt[6]{\left(\frac{abc}{8}\right)^2} = 6\sqrt[3]{\frac{abc}{8}} = 3\sqrt[3]{abc}$$

2. INCENTRE

For incentre we denote $AA_2 = i_a; BB_2 = i_b; CC_2 = i_c$

$$\frac{A_1B}{A_1C} = \frac{c}{b} \Rightarrow \frac{A_1B}{A_1B + A_1C} = \frac{c}{c+b} \Rightarrow \frac{A_1B}{a} = \frac{c}{b+c}$$

$$A_1B = \frac{ac}{b+c}; x = \frac{ac}{b+c}; y = \frac{ba}{c+a}; z = \frac{cb}{a+b}$$

$$i_a = w_a + \frac{x(a-x)}{w_a} = w_a + \frac{\frac{ac}{b+c}(a - \frac{ac}{b+c})}{w_a} =$$

$$= w_a = \frac{ac(ab+ac-ac)}{w_a(b+c)^2} = w_a + \frac{a^2bc}{w_a(b+c)^2}$$

$$i_a = w_a + \frac{a^2bc}{w_a(b+c)^2}; i_b = w_b + \frac{ab^2c}{w_b(c+a)^2}; i_c = w_c + \frac{abc^2}{w_c(a+b)^2}$$

The inequalities (1); (2) ca be written:

$$i_a i_b i_c \geq 8\sqrt{\frac{ac}{b+c} \cdot \frac{cb}{c+a} \cdot \frac{ac}{a+b} \cdot \left(a - \frac{ac}{b+c}\right) \cdot \left(b - \frac{ba}{c+a}\right)\left(c - \frac{ab}{a+b}\right)} =$$

$$= 8\sqrt{\frac{a^2b^2c^2}{(a+b)^2(b+c)^2(c+a)^2} \cdot ab \cdot bc \cdot ca} =$$

$$= 8\sqrt{\frac{a^4b^4c^4}{(a+b)^2(b+c)^2(c+a)^2}} = \frac{8a^2b^2c^2}{(a+b)(b+c)(c+a)}$$

$$i_a + i_b + i_c \geq 6\sqrt[6]{\frac{ac}{b+c} \cdot \frac{cb}{c+a} \cdot \frac{ac}{a+b}\left(a - \frac{ac}{b+c}\right)\left(b - \frac{ba}{c+a}\right)\left(c - \frac{cb}{a+b}\right)} =$$

$$= 6\sqrt[6]{\frac{a^4b^4c^4}{(a+b)^2(b+c)^2(c+a)^2}} = 6\sqrt[3]{\frac{a^2b^2c^2}{(a+b)(b+c)(c+a)}}$$

3. LEMOINE POINT

For Lemoine point we denote $AA_2 = L_a; BB_2 = L_b; CC_2 = L_c; \frac{A_1B}{A_1C} = \frac{c^2}{b^2} \Rightarrow \frac{A_1B}{A_1B + A_1C} = \frac{c^2}{c^2+a^2}$

$$\frac{A_1B}{a} = \frac{c^2}{c^2+b^2} \Rightarrow A_1B = \frac{ac^2}{c^2+b^2}$$

$$x = \frac{ac^2}{c^2+b^2}; y = \frac{ba^2}{a^2+c^2}; z = \frac{cb^2}{c^2+a^2}$$

$$L_a = s_a + \frac{x(a-x)}{s_a} = s_a + \frac{\frac{ac^2}{c^2+b^2}(a - \frac{ac^2}{c^2+b^2})}{s_a} =$$

$$= s_a + \frac{ac^2(ac^2+ab^2-ac^2)}{s_a(b^2+c^2)^2} = s_a + \frac{a^2b^2c^2}{s_a(b^2+c^2)^2}$$

$$L_a = s_a + \frac{a^2b^2c^2}{s_a(b^2+c^2)^2}; L_b = s_b + \frac{a^2b^2c^2}{s_b(c^2+a^2)^2}; L_c = s_c + \frac{a^2b^2c^2}{s_c(a^2+b^2)^2}$$

$$L_aL_bL_c \geq 8\sqrt{\frac{ac^2}{c^2+b^2} \cdot \frac{ba^2}{a^2+c^2} \cdot \frac{cb^2}{c^2+a^2}\left(a - \frac{ac^2}{c^2+b^2}\right)\left(b - \frac{ba^2}{a^2+c^2}\right)\left(c - \frac{cb^2}{b^2+a^2}\right)} =$$

$$= 8\sqrt{\frac{a^3b^3c^3}{(a^2+b^2)^2(b^2+c^2)^2(c^2+a^2)^2} \cdot b^2a \cdot c^2b \cdot a^2c} =$$

$$= \frac{8a^3b^3c^3}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}$$

$$L_a + L_b + L + c \geq 6\sqrt[6]{\frac{ac^2}{c^2+b^2} \cdot \frac{ba^2}{a^2+c^2} \cdot \frac{cb^2}{c^2+a^2}\left(a - \frac{ac^2}{c^2+b^2}\right)\left(b - \frac{ba^2}{a^2+c^2}\right)\left(c - \frac{cb^2}{b^2+a^2}\right)} =$$

$$= 6\sqrt[6]{\frac{a^6b^6c^6}{(a^2+b^2)^2(b^2+c^2)^2(c^2+a^2)^2}} = \frac{6abc}{\sqrt[3]{(a^2+b^2)(b^2+c^2)(c^2+a^2)}}$$

4. CIRCUMCENTRE

For circumcenter we denote

$$AA_2 = O_a; BB_2 = O_b; CC_2 = O_C; \frac{A_1B}{A_1C} = \frac{\sin 2C}{\sin 2B} \Rightarrow \frac{A_1B}{A_1B + A_1C} = \frac{\sin 2C}{\sin 2C + \sin 2B}$$

$$\frac{A_1B}{a} = \frac{\sin 2C}{\sin 2C + \sin 2B} \Rightarrow A_1B = \frac{a\sin 2C}{\sin 2C + \sin 2B}$$

$$x = \frac{a\sin 2C}{\sin 2C + \sin 2B}; y = \frac{b\sin 2A}{\sin 2A + \sin 2C}; z = \frac{c\sin 2B}{\sin 2B + \sin 2A}$$

$$O_a \cdot O_b \cdot O_c \geq 8\sqrt{a^2b^2c^2 \cdot \prod_{cyc} \frac{\sin 2C}{\sin 2C + \sin 2B}\left(1 - \frac{\sin 2C}{\sin 2C + \sin 2B}\right)} =$$

$$= 8abc\sqrt{\prod_{cyc} \frac{\sin 2C \sin 2B}{(\sin 2C + \sin 2B)^2}} =$$

$$= \frac{8abc\sin 2A \sin 2B \sin 2C}{(\sin 2A + \sin 2B)(\sin 2B + \sin 2C)(\sin 2C + \sin 2A)}$$

$$O_a + O_b + O_c \geq 8\sqrt[6]{a^2b^2c^2 \prod_{cyc} \frac{\sin 2B \sin 2C}{(\sin 2B + \sin 2C)^2}} =$$

$$= 8\sqrt[3]{\frac{abc\sin 2A \sin 2B \sin 2C}{(\sin 2A + \sin 2B)(\sin 2B + \sin 2C)(\sin 2C + \sin 2A)}}$$

5. GERGONE POINT

For Gergonne's point we denote

$$AA_2 = G_a, BB_2 = G_b; CC_2 = G_c; \frac{A_1B}{A_1C} = \frac{\frac{1}{s-c}}{\frac{1}{s-b}} = \frac{s-b}{s-c}$$

$$\frac{A_1B}{A_1B + A_1C} = \frac{s-b}{s-b+s-c} \Rightarrow \frac{A_1B}{a} = \frac{s-b}{a} \Rightarrow A_1B = s-b$$

$$x = s-b; y = s-c; z = s-a$$

$$G_a \cdot G_b \cdot G_c \geq 8\sqrt{(s-b)(s-c)(s-a)(a-s+b)(b-s+c)(c-s+a)} =$$

$$= 8\sqrt{\frac{1}{s} \cdot s^2\left(a+b-\frac{a+b+c}{2}\right)\left(b+c-\frac{a+b+c}{2}\right)\left(a+c-\frac{a+b+c}{2}\right)} =$$

$$= 8S\sqrt{\frac{1}{s} \cdot \frac{(a+b-c)(b+c-a)(c+a-b)}{8}} =$$

$$= 4s\sqrt{\frac{(a+b-c)(b+c-a)(c+a-b)}{s}}$$

$$G_a + G_b + G_c \geq 6\sqrt[6]{(s-b)(s-c)(s-a)(a-s+b)(b-s+c)(c-s+a)} =$$

$$= 6\sqrt[6]{\frac{S^2}{s} \cdot \frac{(a+b-c)(b+c-a)(c+a-b)}{8}} =$$

$$= 6\sqrt[6]{\frac{(a+b-c)(b+c-a)(c+a-b)S^2}{8s}}$$

6. NAGEL POINT

For Nagel's point we denote

$$AA_2 = N_a, BB_2 = N_b; CC_2 = N_c; \frac{A_1B}{A_1C} = \frac{s-c}{s-b} \Rightarrow$$

$$\Rightarrow \frac{A_1B}{A_1B+A_1C}=\frac{s-c}{s-c+s-b}\Rightarrow \frac{A_1B}{a}=\frac{s-c}{a}\Rightarrow A_1B=s-c$$

$$x=s-c; y=s-a; z=s-b$$

$$N_a\cdot N_b\cdot N_c\geq 8\sqrt{(s-c)(s-a)(s-c)(a-s+c)(b-s+a)(c-s+b)}=$$

$$=8\sqrt{\frac{1}{s}\cdot S^2\left(a+c-\frac{a+b+c}{2}\right)\left(b+a-\frac{a+b+c}{2}\right)\left(c+b-\frac{a+b+c}{2}\right)}=$$

$$=8\sqrt{\frac{S^2}{s}\cdot\frac{a+c-b}{2}\cdot\frac{b+a-c}{2}\cdot\frac{c+b-a}{2}}=$$

$$=\frac{8S}{2}\sqrt{\frac{(a+c-b)(b+a-c)(c+b-a)}{2s}}=$$

$$=4S\sqrt{\frac{(a+c-b)(b+a-c)(c+b-a)}{2s}}$$

$$N_a+N_b+N_c\geq 6\sqrt[6]{(s-c)(s-a)(s-c)(a-s+c)(b-s+a)(c-s+b)}=$$

$$=6\sqrt[6]{\frac{1}{s}\cdot S^2\cdot\frac{(a+c-b)(b+a-c)(c+b-a)}{8s}}=$$

$$=6\sqrt[6]{\frac{(a+c-b)(b+a-c)(c+b-a)S^2}{8s^2}}$$

7. ORTHOCENTRE

For orthocenter in acute ΔABC we denote:

$$AA_2=H_a, BB_2=H_b, CC_2=H_c$$

$$\frac{A_1B}{A_1C}=\frac{\tan C}{\tan B}\Rightarrow\frac{A_1B}{A_1B+A_1C}=\frac{\tan C}{\tan C+\tan B}\Rightarrow$$

$$\Rightarrow\frac{A_1B}{a}=\frac{\tan C}{\tan C+\tan B}\Rightarrow A_1B=\frac{a\tan C}{\tan C+\tan B}$$

$$x=\frac{a\tan C}{\tan C+\tan B}; y=\frac{b\tan A}{\tan A+\tan C}; z=\frac{c\tan B}{\tan C+\tan A}$$

$$H_aH_bH_c\geq 8\sqrt{\prod_{cyc}\frac{a\tan C}{\tan C+\tan B}\left(a-\frac{a\tan C}{\tan C+\tan B}\right)}=$$

$$=8\sqrt{a^2b^2c^2\cdot\frac{\tan^2 C\cdot\tan^2 B\cdot\tan^2 A}{(\tan C+\tan B)(\tan A+\tan C)(\tan B+\tan A)}}$$

$$=\frac{8abc\tan A\tan B\tan C}{(\tan A+\tan B)(\tan B+\tan C)(\tan C+\tan A)}$$

$$H_a+H_b+H_c\geq 6\sqrt[6]{\frac{a^2b^2c^2\tan^2 A\tan^2 B\tan^2 C}{(\tan A+\tan B)(\tan B+\tan C)(\tan C+\tan A)}}$$

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