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G –centroid of ΔABC

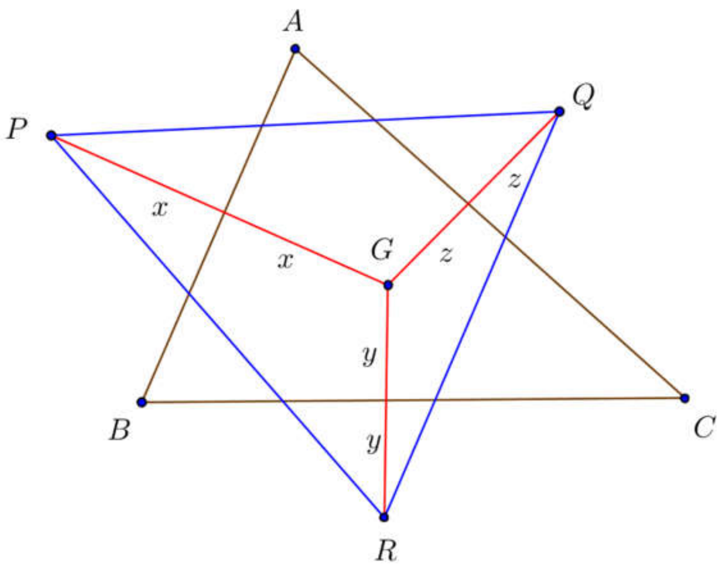
P, Q, R –symmetrical points of G

to BC, CA, AB respectively.

If $\frac{[PQR]}{[ABC]} = 1$, then

ΔABC –equilateral

If $\frac{[PQR]}{[ABC]} = \frac{8}{9}$, then ΔABC –right



Proposed by Thanasis Gakopoulos-Farsala-Greece
Solution 1 by Jose Ferreira Queiroz-Olinda-Brazil, Solution 2 by proposer

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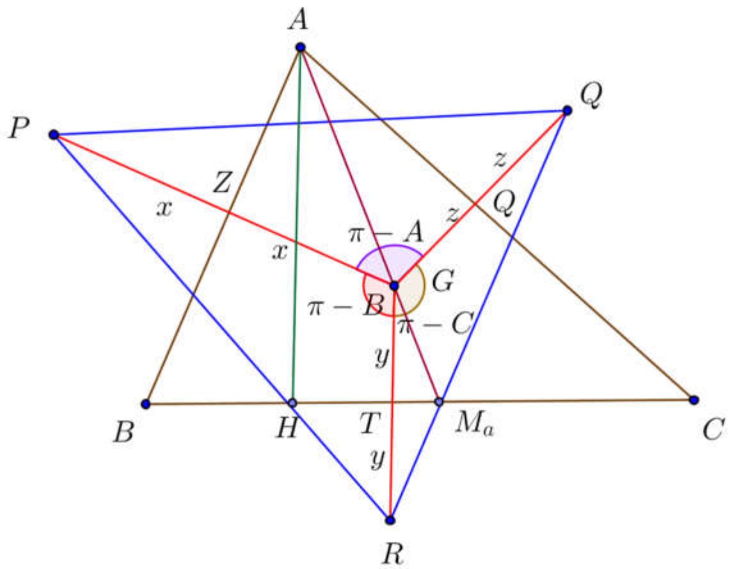

$$AH = h_a, [ABC] = F, h_a = \frac{2F}{a}, h_b = \frac{2F}{b}, h_c = \frac{2F}{c}, sinA = \frac{2F}{bc}, sinB = \frac{2F}{ac},$$
$$sinC = \frac{2F}{ab}, F = \frac{abc}{4R}$$

$\Delta AHT \sim \Delta GTM_a \Rightarrow GT = \frac{h_a}{3}$. In the same way: $cosQ = \frac{h_b}{3}$ and $GZ = \frac{h_c}{3}$.

$$[PQR] = [PGR] + [PGQ] + [RGQ]$$

$$[PQR] = \frac{1}{2}\cdot\frac{2}{3}h_a\cdot\frac{2}{3}h_c sinB + \frac{1}{2}\cdot\frac{2}{3}h_a\cdot\frac{2}{3}h_b sinC + \frac{1}{2}\cdot\frac{2}{3}h_b\cdot\frac{2}{3}h_c sinA$$

$$[PQR] = \frac{2}{9}(h_b h_c sinA + h_a h_c sinB + h_a h_b sinC) =$$

$$= \frac{2}{9}\left(\frac{2F}{b}\cdot\frac{2F}{c}sinA + \frac{2F}{c}\cdot\frac{2F}{a}sinB + \frac{2F}{a}\cdot\frac{2F}{b}sinC\right) =$$

$$= \frac{8F^2}{9}\left(\frac{1}{bc}sinA + \frac{1}{ca}sinB + \frac{1}{ab}sinC\right) = \frac{8F^2}{9}\left(\frac{1}{bc}\cdot\frac{2F}{bc} + \frac{1}{ca}\cdot\frac{2F}{ca} + \frac{1}{ab}\cdot\frac{2F}{ab}\right) =$$

$$\frac{16F^3}{9}\cdot\frac{a^2+b^2+c^2}{a^2b^2c^2} = \frac{16F^3}{9}\cdot\frac{a^2+b^2+c^2}{16R^2F^2} = \frac{F}{9R^2}\cdot(a^2+b^2+c^2)$$

$$i)\ \frac{[PQR]}{[ABC]} = 1 \Rightarrow \frac{F}{9R^2}\cdot(a^2+b^2+c^2) = F \Rightarrow a^2+b^2+c^2 = 9F$$

Euler line: Distance between the circumcenter and the orthocentre triangle:

$$OH^2 = R^2 - \frac{1}{9}(a^2+b^2+c^2)$$

$$\begin{cases} OH^2 = 9R^2 - 9R^2 = 0 \\ GO^2 = R^2 - \frac{1}{9}\cdot 9R^2 = 0 \end{cases} \Rightarrow G = O = H \Rightarrow \Delta ABC \text{ −equilateral.}$$

$$ii)\ \frac{[PQR]}{[ABC]} = \frac{8}{9} \Rightarrow \frac{F}{9R^2}\cdot(a^2+b^2+c^2) = \frac{8}{9}F \Rightarrow a^2+b^2+c^2 = 8R^2$$

$$\text{So, }\begin{cases} OH^2 = 9R^2 - 8R^2 = R^2 \\ GO^2 = R^2 - \frac{1}{9}\cdot 8R^2 = \frac{R^2}{9} \end{cases} \Rightarrow \Delta ABC \text{ −right.}$$

Solution 2 by proposer

Plagiogonal system: $BC \equiv Bx, BA \equiv By, B(0,0), G\left(\frac{a}{3},\frac{a}{3}\right), P(p_1,p_2),$

$$p_1 = \frac{a+2ccosB}{3}, p_2 = -\frac{c}{3}, R(r_1,r_2), r_1 = -\frac{a}{3}, r_2 = \frac{c+2acosB}{3}$$

$$[GPR] = \frac{sinB}{2}\left\|\begin{matrix} 1 & 1 & 1 \\ \frac{a}{3} & r_1 & p_1 \\ \frac{c}{3} & r_2 & p_2 \end{matrix}\right\| = \frac{sinB}{2}\cdot ac\cdot\frac{4}{9}sin^2B$$

$\frac{[GPR]}{[ABC]} = \frac{4}{9}sin^2B; (1)$, similarly $\frac{[GPQ]}{[ABC]} = \frac{4}{9}sin^2C; (2), \frac{[GQR]}{[ABC]} = \frac{4}{9}sin^2A; (3)$

From (1),(2),(3), it follows that: $\frac{[PQR]}{[ABC]} = \frac{4}{9}(sin^2A + sin^2B + sin^2C); (4)$

$$i)\ \frac{[PQR]}{[ABC]} = 1 \Rightarrow sin^2A + sin^2B + sin^2C = \frac{9}{4} \Rightarrow 2 + 2cosAcosBcosC = \frac{9}{4} \Rightarrow$$

$$cosAcosBcosC = \frac{1}{8} \Rightarrow \Delta ABC - \text{equilateral}$$

$$ii)\ \frac{[PQR]}{[ABC]} = \frac{8}{9} \Rightarrow sin^2A + sin^2B + sin^2C = \frac{8}{9} \cdot \frac{9}{4} \Rightarrow 2 + 2cosAcosBcosC = 2 \Rightarrow$$

$$cosAcosBcosC = 0 \Rightarrow \Delta ABC - \text{right.}$$

Note by editor:
Many thanks to Florică Anastase-Romania for typed solutions.