IDENTITIES AND INEQUALITIES IN CYCLIC QUADRILATERALS

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Abstract. In this paper are proved several metric identities in cyclic quadrilaterals.

Notations:
If $ABCD$ is a cyclic quadrilateral denote:

$$AB = a; BC = b; CD = c; DA = d; AC = e; BD = f; \Delta \text{ - area; } s \text{ - semiperimeter;}$$

$$r_a, r_b, r_c, r_d \text{ - exradii; } R \text{ - circumradii.}$$

By Bretschneider's formula:

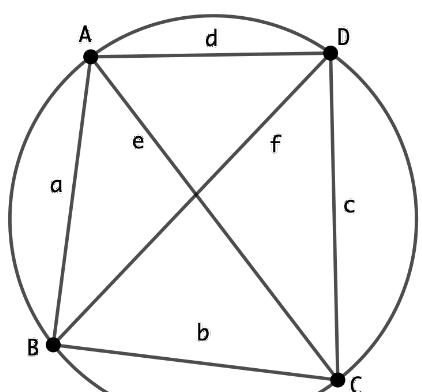
$$\Delta = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\left(\frac{A+C}{2}\right)} \tag{1}$$

$$A + C = B + D = \pi; \sin A = \sin C; \sin B = \sin D; \cos A - \cos C; \cos B = -\cos D$$

Replace $A + C = \pi$ in (1) and we obtain:

$$\Delta = \sqrt{(s-a)(s-b)(s-c)(s-d)} \tag{2}$$

which is called Brahmagupta's formula.



Ptolemy's theorem states that:

$$ef = ac + bd \tag{3}$$

Proposition 1:
In any $ABCD$ cyclic quadrilateral holds:

$$\cos A = \frac{a^2 - b^2 - c^2 + d^2}{2(ad + bc)} = -\cos C \tag{4}$$

Proof. By the law of cosine in $\Delta ABD; \Delta BCD$:

$$f^2 = a^2 + d^2 - 2ad\cos A = b^2 + c^2 - 2bc\cos C$$

$$a^2 - b^2 - c^2 + d^2 = 2(ad\cos A - bc\cos(\pi - A))$$

$$2\cos A(ad + bc) = a^2 - b^2 - c^2 + d^2$$

$$\cos A = \frac{a^2 - b^2 - c^2 + d^2}{2(ad + bc)}$$

□

Proposition 2:
In any $ABCD$ cyclic quadrilateral holds:

$$\sin A = \sin C = \frac{2\Delta}{ad + bc} \tag{5}$$

$$\sin B = \sin D = \frac{2\Delta}{ab + cd} \tag{6}$$

Proof.

$$\Delta = S[ABD] + S[BCD] = \frac{1}{2}ad\sin A + \frac{1}{2}bc\sin C \Rightarrow$$

$$2\Delta = ad\sin A + bc\sin(\pi - A)$$

$$2\Delta = (ad + bc)\sin A \Rightarrow \sin A = \frac{2S}{ad + bc} = \sin C$$

$$\Delta = S[ABC] + S[ACD] = \frac{1}{2}ab\sin B + \frac{1}{2}cd\sin D$$

$$2\Delta = ab\sin B + cd\sin(\pi - B)$$

$$2\Delta = \sin B(ab + cd) \Rightarrow \sin B = \frac{2\Delta}{ab + cd} = \sin D$$

□

Proposition 3:
In any $ABCD$ cyclic quadrilateral holds:

$$\cos^2\frac{A}{2} = \frac{(s-b)(s-c)}{ad + bc} = \sin^2\frac{C}{2}; \cos^2\frac{B}{2} = \frac{(s-c)(s-d)}{ba + cd} = \sin^2\frac{B}{2}$$

$$\cos^2\frac{C}{2} = \frac{(s-d)(s-a)}{cb + da} = \sin^2\frac{A}{2}; \cos^2\frac{D}{2} = \frac{(s-a)(s-b)}{dc + ab} = \sin^2\frac{B}{2} \tag{7}$$

Proof.

$$\cos^2\frac{A}{2} = \frac{1 + \cos A}{2} = \frac{1 + \frac{a^2 - b^2 - c^2 + d^2}{2(ad + bc)}}{2} =$$

$$= \frac{2(ad + bc) + a^2 - b^2 - c^2 + d^2}{4(ad + bc)} = \frac{(a + d)^2 - (b - c)^2}{4(ad + bc)} =$$

$$= \frac{(a + d - b + c)(a + d + b - c)}{4(ad + bc)} = \frac{(2s - 2b)(2s - 2c)}{4(ad + bc)} = \frac{(s-b)(s-c)}{ad + bc}$$

□

Proposition 4:
In any $\Delta ABCD$ cyclic quadrilateral holds:

$$\tan^2\frac{A}{2}=\frac{(s-a)(s-d)}{(s-b)(s-c)}=\cot^2\frac{C}{2};\tan^2\frac{B}{2}=\frac{(s-b)(s-a)}{(s-c)(s-d)}=\cot^2\frac{D}{2}$$

$$\tan^2\frac{C}{2}=\frac{(s-c)(s-b)}{(s-d)(s-a)}=\cot^2\frac{A}{2};\tan^2\frac{D}{2}=\frac{(s-d)(s-c)}{(s-a)(s-b)}=\cot^2\frac{B}{2} \tag{8}$$

Proof.

$$\tan^2\frac{A}{2}=\frac{\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}}=\frac{\frac{(s-d)(s-a)}{ad+bc}}{\frac{(s-b)(s-c)}{ad+bc}}=\frac{(s-a)(s-d)}{(s-b)(s-c)}=\tan^2\left(\frac{\pi-C}{2}\right)=\cot^2\frac{C}{2}$$

□

Proposition 5: (GIRARD'S IDENTITY)
In any $ABCD$ cyclic quadrilateral holds:

$$(ab+cd)(ac+bd)(ad+bc)=16R^2\Delta^2 \tag{9}$$

Proof. In $\Delta ABD:\frac{f}{2\sin A}=R$; In $\Delta ABC:\frac{e}{2\sin B}=R$
By proposition (2):

$$\frac{f}{2\cdot\frac{2\Delta}{ad+bc}}=R\Rightarrow f=\frac{4\Delta R}{ad+bc} \tag{10}$$

$$\frac{e}{2\cdot\frac{2\Delta}{ab+cd}}=R\Rightarrow e=\frac{4\Delta R}{ab+cd} \tag{11}$$

By multiplying (10); (11):

$$ef=\frac{16\Delta^2R^2}{(ad+bc)(ab+cd)}$$

By Ptolemy's theorem: $ef=ac+bd$, hence:

$$ac+bd=\frac{16R^2\Delta^2}{(ad+bc)(ab+cd)}$$

$$(ab+cd)(ac+bd)(ad+bc)=16R^2\Delta^2$$

□

Proposition 6 (circumradii)
In any $\Delta ABCD$ cyclic quadrilateral holds:

$$R=\frac{\sqrt{(ab+cd)(ac+bd)(ad+bc)}}{4\Delta} \tag{12}$$

Proof. By Girard's identity:

$$(ab+cd)(ac+bd)(ad+bc)=16R^2\Delta^2$$

$$R^2=\frac{(ab+cd)(ac+bd)(ad+bc)}{16\Delta^2}$$

$$R=\frac{\sqrt{(ab+cd)(ac+bd)(ad+bc)}}{4\Delta}$$

□

Proposition 7 (diagonals)
In any $ABCD$ cyclic quadrilateral holds:

$$e=\sqrt{\frac{(ad+bc)(ac+bd)}{ab+cd}};f=\sqrt{\frac{(ab+cd)(ac+bd)}{ad+bc}}$$

Proof. By (10); (11):

$$e=\frac{4\Delta R}{ab+cd}\overset{(9)}{=}\frac{1}{ab+cd}\sqrt{(ab+cd)(ac+bd)(ad+bc)}=$$

$$=\sqrt{\frac{(ad+bc)(ac+bd)}{ab+cd}}$$

$$f=\frac{4\Delta R}{ad+bc}\overset{(9)}{=}\frac{1}{ad+bc}\sqrt{(ab+cd)(ac+bd)(ad+bc)}=$$

$$=\sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}$$

The circles tangent to a side of the cyclic quadrilateral and tangent to the extensions of its two other sides has radii r_a, r_b, r_c, r_d called exradii. □

Proposition 8 (exradii)
In any $ABCD$ cyclic quadrilateral holds:

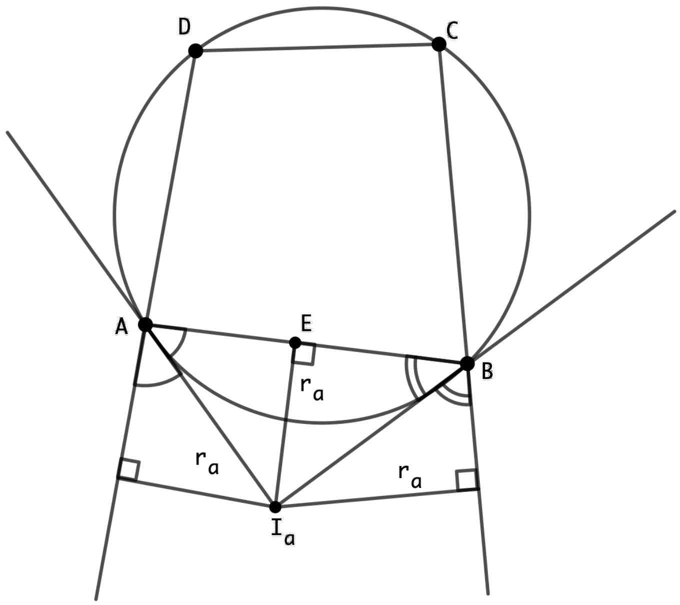
(13) $$r_a=\frac{a}{\tan\frac{A}{2}+\tan\frac{B}{2}}=\frac{\Delta a}{(s-a)(a-c)}$$

$$r_b=\frac{b}{\tan\frac{B}{2}+\tan\frac{C}{2}}=\frac{\Delta b}{(s-b)(b+d)}$$

$$r_c=\frac{c}{\tan\frac{C}{2}+\tan\frac{D}{2}}=\frac{\Delta c}{(s-c)(a+c)}$$

$$r_d=\frac{d}{\tan\frac{\Delta}{2}+\tan\frac{A}{2}}=\frac{\Delta d}{(s-d)(b+d)}$$

Proof. Denote I_a - centre of excircle tangent to AB.



$$\tan(\angle I_aAE) = \frac{I_aE}{AE} = \frac{r_a}{AE} \tag{14}$$

$$\tan(\angle I_aAE) = \tan\left(\frac{\pi - A}{2}\right) = \cot\frac{A}{2} \tag{15}$$

By (14); (15):

$$\frac{r_a}{AE} = \cot\frac{A}{2} \Rightarrow AE = r_a \tan\frac{A}{2} \tag{16}$$

$$\tan(\angle I_aBE) = \frac{I_aE}{BE} = \frac{r_a}{BE} \tag{17}$$

$$\tan(\angle I_aBE) = \tan\left(\frac{\pi - B}{2}\right) = \cot\frac{B}{2} \tag{18}$$

By (17); (18):

$$\frac{r_a}{BE} = \cot\frac{B}{2} \Rightarrow BE = r_a = \tan\frac{B}{2} \tag{19}$$

$$a = AB = AE + EB \overset{(16);(19)}{=} r_a \tan\frac{A}{2} + r_a \tan\frac{B}{2} =$$

$$= r_a\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right) \Rightarrow r_a = \frac{a}{\tan\frac{A}{2} + \tan\frac{B}{2}}$$

$$r_a = \frac{a}{\tan\frac{A}{2} + \tan\frac{B}{2}} = \frac{a\Delta}{\Delta\tan\frac{A}{2} + \Delta\tan\frac{B}{2}} =$$

$$\overset{(2)}{=} \frac{a\Delta}{\sqrt{(s-a)(s-b)(s-c)(s-d)} \cdot \sqrt{\frac{(s-a)(s-d)}{(s-b)(s-c)}} + \sqrt{(s-a)(s-b)(s-c)(s-d)} \cdot \sqrt{\frac{(s-b)(s-a)}{(s-c)(s-d)}}} =$$

$$= \frac{a\Delta}{(s-a)(s-d) + (s-b)(s-a)} = \frac{a\Delta}{(s-a)(2s-b-d)} = \frac{a\Delta}{(s-a)(a+c)}$$

□

Proposition 9:
In any $ABCD$ cyclic quadrilateral holds:

$$\frac{r_a^2}{a^3}+\frac{r_b^2}{b^3}+\frac{r_c^2}{c^3}+\frac{r_d^2}{d^3}\geq\frac{2}{s}$$

Proof. By (13):

$$\frac{r_a}{a}=\frac{\Delta}{(s-a)(a+c)};\frac{r_c}{c}=\frac{\Delta}{(s-c)(a+c)}$$

By adding:

$$\frac{r_a}{a}+\frac{r_c}{c}=\frac{\Delta}{a+c}\Big(\frac{1}{s-a}+\frac{1}{s-c}\Big)=$$

$$=\frac{\Delta}{a+c}\cdot\frac{s-c+s-a}{(s-a)(s-c)}=\frac{\Delta}{a+c}\cdot\frac{b+d}{(s-a)(s-c)} \tag{20}$$

Analogous:

$$\frac{r_b}{b}+\frac{r_d}{d}=\frac{\Delta}{b+c}\cdot\frac{a+c}{(s-a)(s-c)} \tag{21}$$

By adding (20); (21):

$$\frac{r_a}{a}+\frac{r_b}{b}+\frac{r_c}{c}+\frac{r_d}{d}=\frac{\Delta}{a+c}\cdot\frac{b+d}{(s-a)(s-c)}+\frac{\Delta}{b+d}\cdot\frac{a+c}{(s-b)(s-c)}\geq$$

$$\overset{\text{AM-GM}}{=}\Delta\cdot 2\sqrt{\frac{(b+d)(a+c)}{(a+c)(b+d)(s-a)(s-b)(s-c)(s-d)}}=$$

$$=\Delta\cdot\frac{2}{\Delta}=2 \tag{21}$$

$$\frac{r_a}{a^2}+\frac{r_b}{b^2}+\frac{r_c}{c^2}+\frac{r_d}{d^2}=\frac{(\frac{r_a}{a})^2}{a}+\frac{(\frac{r_b}{b})^2}{b}+\frac{(\frac{r_c}{c})^2}{c}+\frac{(\frac{r_d}{d})^2}{d}\geq$$

$$\overset{\text{BERGSTRÖM}}{\geq}\frac{(\frac{r_a}{a}+\frac{r_b}{b}+\frac{r_c}{c}+\frac{r_d}{d})^2}{a+b+c+d}\overset{(21)}{\geq}\frac{2^2}{2s}=\frac{2}{s}$$

Equality holds for $a=b=c=d$ ($ABCD$ - square) □

Proposition 10:
In any $ABCD$ cyclic quadrilateral holds:

$$\frac{a}{r_a^2}+\frac{b}{r_b^2}+\frac{c}{r_c^2}+\frac{d}{r_d^2}\geq\frac{32}{s}$$

Proof. By (13):

$$\frac{a}{r_a}=\tan\frac{A}{2}+\tan\frac{B}{2};\frac{b}{r_b}=\tan\frac{B}{2}+\tan\frac{C}{2}$$

$$\frac{c}{r_c}=\tan\frac{C}{2}+\tan\frac{D}{2};\frac{d}{r_d}=\tan\frac{D}{2}+\tan\frac{A}{2}$$

By adding:

$$\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}+\frac{d}{r_d}=2\Big(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}+\tan\frac{D}{2}\Big) \tag{22}$$

Let be $f:(0,\frac{\pi}{2})\to\mathbb{R}; f(x)=\tan\frac{x}{2}; f'(x)=\frac{1}{2\cos^2\frac{x}{2}}$

$$f''(x)=\frac{-(2\cos^2\frac{x}{2})'}{\cos^4\frac{x}{2}}=\frac{4\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^4\frac{x}{2}}=\frac{2\sin x}{\cos^3\frac{x}{2}}>0; f\text{ - convexe}$$

By Jensen's inequality:

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}+\tan\frac{D}{2}\geq 4\tan\Big(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}+\frac{D}{2}}{2}\Big)=$$

$$=4\tan\Big(\frac{A+B+C+D}{8}\Big)=4\tan\Big(\frac{2\pi}{8}\Big)=4\tan\frac{\pi}{4}=4 \tag{23}$$

By (22); (23):

$$\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}+\frac{d}{r_d}\geq 2\cdot 4=8 \tag{24}$$

$$\frac{a}{r_a^2}+\frac{b}{r_b^2}+\frac{c}{r_c^2}+\frac{d}{r_d^2}=\frac{(\frac{a}{r_a})^2}{a}+\frac{(\frac{b}{r_b})^2}{b}+\frac{(\frac{c}{r_c})^2}{c}+\frac{(\frac{d}{r_d})^2}{d}\geq$$

$$\overset{\text{BERGSTRÖM}}{\geq}\frac{(\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}+\frac{d}{r_d})^2}{a+b+c+d}\overset{(24)}{\geq}\frac{8^2}{2s}=\frac{32}{s}$$

Equality holds for $a=b=c=d$ ($ABCD$ - square) □

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