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Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function at $x=0$ such that:

$$f(x)+f\left(\frac{x}{x+1}\right)=x^2,\forall x\in\mathbb{R}.$$

Then find the value of $f(1)$.

Proposed by Surjeet Singhania-India

Solution by Ravi Prakash-New Delhi-India

$$f(0)+f(0)=0\Rightarrow f(0)=0$$

$$f(x)+f\left(\frac{x}{x+1}\right)=x^2\Rightarrow f\left(\frac{1}{n}\right)+f\left(\frac{1}{n+1}\right)=\frac{1}{n^2},\forall n\in\mathbb{N}$$

$$\sum_{k=1}^{n}(-1)^{k-1}\left[f\left(\frac{1}{k}\right)+f\left(\frac{1}{k+1}\right)\right]=\sum_{k=1}^{n}(-1)^{k-1}\cdot\frac{1}{k^2}\Rightarrow$$

$$f(1)+(-1)^{n-1}f\left(\frac{1}{n+1}\right)=\sum_{k=1}^{n}(-1)^{k-1}\cdot\frac{1}{k^2};(1)$$

As f –continuous at 0, $f\left(\frac{1}{k+1}\right)\to f(0)=0$, as $k\to\infty$

$$\Rightarrow(-1)^{k-1}f\left(\frac{1}{k+1}\right)\to 0,\text{ as } k\to\infty$$

Taking limit as $n\to\infty$ on as$k\to\infty$ in (1), we get:

$$f(1)+0=\sum_{k=1}^{\infty}(-1)^{n-1}\cdot\frac{1}{n^2}=\frac{\pi^2}{12}$$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solution.