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FEW OUTSTANDING LIMITS

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Problem 1.

If $n,k\in\mathbb{N}, n\ge k$ and $f_k:\mathbb{R}\to\left[0,\frac{n}{n+k-1}\right]$ continuous function. Prove that:

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}f(k)\cdot\sqrt[n+k-1]{1-f(k)+\frac{1-k}{n}f(k)}\right)<log2$$

Solution.

$$\frac{n}{n+k}=\frac{(n+k-1)f(k)+n-(n+k-1)f(k)}{n+k}\overset{AM-GM}{\ge}$$

$$\ge\sqrt[n+k]{f^{n+k-1}(k)\big(n-(n+k-1)f(k)\big)}\Leftrightarrow$$

$$f^{n+k-1}(k)\big(n-(n+k-1)f(k)\big)\le\left(\frac{n}{n+k}\right)^{n+k-1}\cdot\frac{n}{n+k}\Leftrightarrow$$

$$f(k)\sqrt[n+k-1]{\frac{n+k}{n}\big(n-(n+k-1)f(k)\big)}\le\frac{n}{n+k}$$

$$\frac{f(k)}{n}\cdot\sqrt[n+k-1]{\frac{n+k}{n}\big(n-(n+k-1)f(k)\big)}\le\frac{1}{n}\cdot\frac{n}{n+k}\Leftrightarrow$$

$$\frac{1}{n}\sum_{k=1}^{n}f(k)\cdot\sqrt[n+k-1]{1-f(n)+\frac{1-k}{n}f(n)}\le$$

$$\le\frac{1}{n}\sum_{k=1}^{n}f(k)\cdot\sqrt[n+k-1]{\frac{n+k}{n}\big(n-(n+k-1)f(k)\big)}\le\frac{1}{n}\cdot\sum_{k=1}^{n}\frac{n}{n+k}=$$

$$\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=1}^{n}\frac{n}{n+k}=\int_0^1\frac{1}{x+1}dx=log2$$

Therefore,

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}f(k)\cdot\sqrt[n+k-1]{1-f(k)+\frac{1-k}{n}f(k)}\right)<log2$$

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Problem 2.

If $(a_n)_{n\geq 1}$ is sequence of real numbers such that

$0 < a_k < \frac{n^2}{n^2+k^2-1}\ \forall n,k \in \mathbb{N}$ then prove:

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k \cdot \sqrt[n^2+k^2-1]{\left(1+\frac{k^2}{n^2}\right)\left(1-\left(1-\frac{1}{n^2}+\frac{k^2}{n^2}\right)a_k\right)}\right) < \frac{\pi}{4}$$

Solution.

$$\frac{n^2}{n^2+k^2} = \frac{(n^2+k^2-1)a_k + n^2 - (n^2+k^2-1)a_k}{n^2+k^2} \overset{AM-GM}{\geq}$$

$$\geq \sqrt[n^2+k^2]{a_k^{n^2+k^2-1}\left(n^2-(n^2+k^2-1)a_k\right)} \Rightarrow$$

$$a_k^{n^2+k^2-1}\left(n^2-(n^2+k^2-1)a_k\right) \leq \left(\frac{n^2}{n^2+k^2}\right)^{n^2+k^2-1} \cdot \frac{n^2}{n^2+k^2} \Leftrightarrow$$

$$a_k \sqrt[n^2+k^2-1]{\frac{\left(n^2-(n^2+k^2-1)a_k\right)(n^2+k^2)}{n^2}} \leq \frac{n^2}{n^2+k^2} \Leftrightarrow$$

$$\frac{a_k}{n} \cdot \sqrt[n^2+k^2-1]{\left(1+\frac{k^2}{n^2}\right)\left(1-\left(1-\frac{1}{n^2}+\frac{k^2}{n^2}\right)a_k\right)} \leq \frac{1}{n} \cdot \frac{n^2}{n^2+k^2} \Leftrightarrow$$

$$\frac{1}{n}\sum_{k=1}^{n} a_k \cdot \sqrt[n^2+k^2-1]{\left(1+\frac{k^2}{n^2}\right)\left(1-\left(1-\frac{1}{n^2}+\frac{k^2}{n^2}\right)a_k\right)} \leq \frac{1}{n} \cdot \sum_{k=1}^{n} \frac{n^2}{n^2+k^2}$$

$$= \frac{1}{n} \cdot \sum_{k=1}^{n} \frac{1}{1+\left(\frac{k}{n}\right)^2}$$

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k \cdot \sqrt[n^2+k^2-1]{\left(1+\frac{k^2}{n^2}\right)\left(1-\left(1-\frac{1}{n^2}+\frac{k^2}{n^2}\right)a_k\right)}\right) <$$

$$< \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=1}^{n} \frac{1}{1+\left(\frac{k}{n}\right)^2} = \int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$$

Therefore,

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\cdot\sqrt[n^2+k^2-1]{\left(1+\frac{k^2}{n^2}\right)\left(1-\left(1-\frac{1}{n^2}+\frac{k^2}{n^2}\right)a_k\right)}\right)<\frac{\pi}{4}$$

Problem 3.

Find:

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\int_{k-1}^{k}tan^{-1}\left(\frac{n(x-k)}{kx+n^2}\right)dx\right)$$

Solution.

$$\sum_{k=1}^{n}\int_{k-1}^{k}tan^{-1}\left(\frac{n(x-k)}{kx+n^2}\right)dx=\sum_{k=1}^{n}\int_{k-1}^{k}tan^{-1}\left(\frac{x-k}{\frac{k}{n}x+n}\right)dx\overset{x\to nx}{=}$$

$$=n\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}tan^{-1}\left(\frac{x-\frac{k}{n}}{1+\frac{k}{n}x}\right)=n\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(tan^{-1}x-tan^{-1}\left(\frac{k}{n}\right)\right)dx=$$

$$=n\left(\int_0^1 tan^{-1}x-\frac{1}{n}\sum_{k=1}^{n}tan^{-1}\left(\frac{k}{n}\right)\right)$$

Let $f:\mathbb{R}\to\mathbb{R}, f(x)=tan^{-1}x, f''(x)<0, \forall x\in\mathbb{R}\Rightarrow f'\searrow$ **and let** $x\in\left[\frac{k-1}{n},\frac{k}{n}\right)\overset{MVT}{\longrightarrow}$

$\exists c\in\left(\frac{k-1}{n},\frac{k}{n}\right)$ **such that** $\frac{f(x)-f\left(\frac{k}{n}\right)}{x-\frac{k}{n}}=f'(c)\Rightarrow$

$$f'\left(\frac{k-1}{n}\right)>\frac{f(x)-f\left(\frac{k}{n}\right)}{x-\frac{k}{n}}>f'\left(\frac{k}{n}\right)|\cdot\left(x-\frac{k}{n}\right)<0\to$$

$$\left(x-\frac{k}{n}\right)f'\left(\frac{k-1}{n}\right)<f(x)-f\left(\frac{k}{n}\right)<\left(x-\frac{k}{n}\right)f'\left(\frac{k}{n}\right)$$

$$-\frac{1}{2n^2}\cdot f'\left(\frac{k-1}{n}\right)\le\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(tan^{-1}x-tan^{-1}\left(\frac{k}{n}\right)\right)\le-\frac{1}{2n^2}f'\left(\frac{k}{n}\right), n\ge 2, k\in\{1,2,\dots,n\}$$

Thus,

$$-\frac{1}{2n}\cdot\sum_{k=1}^{n} f'\left(\frac{k-1}{n}\right) \le n\left(\int_0^1 tan^{-1}x - \frac{1}{n}\sum_{k=1}^{n} tan^{-1}\left(\frac{k}{n}\right)\right) \le -\frac{1}{2n}\cdot\sum_{k=1}^{n} f'\left(\frac{k}{n}\right)$$

$$\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=1}^{n} f'\left(\frac{k-1}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=1}^{n} f'\left(\frac{k}{n}\right) = \int_0^1 f'(x)\,dx = \frac{\pi}{4}$$

Therefore,

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\int_{k-1}^{k} tan^{-1}\left(\frac{n(x-k)}{kx+n^2}\right)dx\right) =$$

$$= \lim_{n\to\infty} n\left(\int_0^1 tan^{-1}x - \frac{1}{n}\sum_{k=1}^{n} tan^{-1}\left(\frac{k}{n}\right)\right) = -\frac{\pi}{8}$$

REFERENCE:

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