FEW BEAUTIFUL CHALLENGING FUNCTIONAL EQUATIONS

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Abstract. In this paper we present some certain results on functional equations.

Theorem 1. All injective functions $f:\mathbb{R}\to\mathbb{R}$ which verify $f(0)\neq\frac{1}{b}$ and $f(f(x)y^3)+ax^9y^9=bf(x^3)f(y^3), \forall x,y\in\mathbb{R}$, where $a>0, b>1$ are functions

$$f_1,f_2:\mathbb{R}\to\mathbb{R}, f_1(x)=\sqrt{\frac{a}{b-1}}\cdot x^3, f_2(x)=-\sqrt{\frac{a}{b-1}}\cdot x^3.$$

Proof. If $x=y=0\Rightarrow f(0)=bf^2(0)\Rightarrow f(0)(bf(0)-1)=0$ and since $f(0)\neq\frac{1}{b}$ yields that $f(0)=0$.
For any $x,y\in\mathbb{R}^*$ by the statement we have $f(f(x)y^3)+ax^9y^9=bf(x^3)f(y^3)$, $\forall x,y\in\mathbb{R}$ and $f(f(y)x^3)+ay^9x^9=bf(y^3)f(x^3)$, then $f(f(x)y^3)=f(f(y)x^3)$, $\forall x,y\in\mathbb{R}^*$ thus by injectivity of f we deduce that
$f(x)y^3=f(y)x^3\Leftrightarrow\frac{f(x)}{x^3}=\frac{f(y)}{y^3},\forall x,y\in\mathbb{R}^*$.
So, $\frac{f(x)}{x^3}=\frac{f(1)}{1^3}\Rightarrow f(x)=f(1)\cdot x^3,\forall x\in\mathbb{R}^*$ and denoting $c=f(1)\in\mathbb{R}^*$, it results $f(x)=cx^3,\forall x\in\mathbb{R}^*$, so by the statement we obtain

$$f(cx^3y^3)+ax^9y^9=b\cdot cx^9\cdot cy^9\Leftrightarrow c^2x^9y^9+ax^9y^9=bc^2x^9y^9\Leftrightarrow a=c^2(b-c)\Leftrightarrow$$

$$\Leftrightarrow c^2=\frac{a}{b-1}\Rightarrow c=\pm\sqrt{\frac{a}{b-1}}$$

Hence, we obtain the functions $f_1,f_2:\mathbb{R}\to\mathbb{R}, f_1(x)=\sqrt{\frac{a}{b-1}}\cdot x^3$,
$f_2(x)=-\sqrt{\frac{a}{b-1}}\cdot x^3$. □

Theorem 2. Let m be a positive integer number, $a,b,c>0$ and d a real number. All the functions $f:\mathbb{R}\to\mathbb{R}$ which verify $af(x)+bf([x])+cf(\{x\})=dx^m,\forall x\in\mathbb{R}$ (where we denote $[x]$, the integer part of x, respective with $\{x\}$ the fractional part of x) are the following functions:

$$f(x)=\frac{d}{a}\Big(x^m-\frac{b}{a+b}[x]^m-\frac{c}{a+c}\{x\}^m\Big)$$

Proof. If $x=0\Rightarrow f(0)=0$.

$$\text{If } x=[x]\Rightarrow af([x])+bf([x])+cf(0)=d[x]^m\Rightarrow f([x])=\frac{d}{a+b}[x]^m,\forall x\in\mathbb{R}$$

If $x=\{x\}\Rightarrow af(\{x\})+bf(0)+cf(\{x\})=d\{x\}^m\Rightarrow f(\{x\})=\dfrac{d}{a+c}\{x\}^m,\forall x\in\mathbb{R}$
Then, by the statement we have $af(x)+\frac{bd}{a+b}[x]^m+\frac{cd}{a+c}\{x\}^m=dx^m,\forall x\in\mathbb{R}$.
Hence $f(x)=\frac{d}{a}\Big(x^m-\frac{b}{a+b}[x]^m-\frac{c}{a+c}\{x\}^m\Big)$ □

Theorem 3. All the functions $f : \mathbb{R} \to \mathbb{R}$ which verify $f(x)+f([x])+f(\{x\}) = ax^m, \forall x, a \in \mathbb{R}$, and m is a positive integer number (where we denote $[x], \{x\}$ the integer part of x, respective the fractional part of x) are $f(x) = a\big(x^m - \frac{1}{2}[x]^m - \frac{1}{2}\{x\}^m\big)$.

Proof. If $x = 0 \Rightarrow f(0) = 0$.

$$\text{If } x = [x] \Rightarrow f([x]) + f([x]) + f(0) = a[x]^m \Rightarrow f([x]) = \frac{a}{2}[x]^m, \forall x \in \mathbb{R}.$$

$$\text{If } x = \{x\} \Rightarrow f(\{x\}) + f(0) + f(\{x\}) = a\{x\}^m \Rightarrow f(\{x\}) = \frac{a}{2}\{x\}^m, \forall x \in \mathbb{R}.$$

Then, by the statement we have $f(x) + \frac{a}{2}([x]^m + \{x\}^m) = ax^m, \forall x \in \mathbb{R}$.
Hence, $f(x) = a\Big(x^m - \frac{1}{2}[x]^m - \frac{1}{2}\{x\}^m\Big)$. □

Theorem 4. Let $a > 0, b > 1$ and $n \in \mathbb{N}$. All injective functions $f : \mathbb{R} \to \mathbb{R}, f(0) \neq \frac{1}{b}$ which verify $f(f(x))y^{2n+1})+a(xy)^{(2n+1)^2} = bf(x^{2n+1})f(y^{2n+1})$, $\forall x, y \in \mathbb{R}$ are the functions

$$f_1, f_2 : \mathbb{R} \to \mathbb{R}, f_1(x) = \sqrt{\frac{a}{b-1}} \cdot x^{2n+1}, f_2(x) = -\sqrt{\frac{a}{b-1}} \cdot x^{2n+1}$$

Proof. If $x = y = 0 \Rightarrow f(0) = bf^2(0) \Rightarrow f(0)(bf(0) - 1) = 0$ and since $f(0) \neq \frac{1}{b}$, it results $f(0) = 0$.
For any $x, y \in \mathbb{R}^*$ by statement we have:

$$f(f(x)y^{2n+1}) + a(xy)^{(2n+1)^2} = bf(x^{2n+1})f(y^{2n+1}), \forall x, y \in \mathbb{R} \text{ and}$$

$$f(f(y)x^{2n+1}) + a(xy)^{(2n+1)^2} = bf(y^{2n+1})f(x^{2n+1}), \forall x, y \in \mathbb{R} \text{ then}$$

$f(f(x)y^{2n+1}) = f(f(y)x^{2n+1}), \forall x, y \in \mathbb{R}^*$, and by injectivity of f we deduce that

$$f(x)y^{2n+1} = f(y)x^{2n+1} \Leftrightarrow \frac{f(x)}{x^{2n+1}} = \frac{f(y)}{y^{2n+1}}, \forall x, y \in \mathbb{R}^*.$$

So, $\frac{f(x)}{x^{2n+1}} = \frac{f(1)}{1^{2n+1}} \Rightarrow f(x) = f(1) \cdot x^{2n+1}, \forall x \in \mathbb{R}^*$, and if we denote $c = f(1) \in \mathbb{R}^*$, yields $f(x) = cx^{2n+1}, \forall x \in \mathbb{R}^*$, then by the statement we obtain

$$f(cx^{2n+1}y^{2n+1}) + ax^{(2n+1)^2}y^{(2n+1)^2} = b \cdot cx^{(2n+1)^2} \cdot cy^{(2n+1)^2} \Leftrightarrow$$

$$\Leftrightarrow c^2(xy)^{(2n+1)^2} + a(xy)^{(2n+1)^2} = bc^2(xy)^{(2n+1)^2} \Leftrightarrow a = c^2(b - c) \Leftrightarrow$$

$$\Leftrightarrow c^2 = \frac{a}{b-1} \Rightarrow c = \pm\sqrt{\frac{a}{b-1}}.$$

We obtain the functions:

$$f_1, f_2 : \mathbb{R} \to \mathbb{R}, f_1(x) = \sqrt{\frac{a}{b-1}} \cdot x^{2n+1}, f_2(x) = -\sqrt{\frac{a}{b-1}} \cdot x^{2n+1}.$$

□

Theorem 5. Let $a > 0, b > 1$ and injective function $f : \mathbb{R} \to \mathbb{R}$ which verify $f(0) = 0$. All injective functions $g : \mathbb{R} \to \mathbb{R}, g(0) \neq \frac{1}{b}$ which satisfy:

$$g(g(x)f(y)) + af(x)f(y) = bg(f(x))g(f(y)), \forall x, y \in \mathbb{R}$$

are the functions:

$$g_1, g_2 : \mathbb{R} \to \mathbb{R}, g_1(x) = \sqrt{\frac{a}{b-1}} \cdot f(x), g_2(x) = -\sqrt{\frac{a}{b-1}} \cdot f(x)$$

Proof. If $x = y = 0$, we have $g(g(0)f(0)) + af(0)f(0) = bg(f(0))g(f(0)) \Leftrightarrow$
$\Leftrightarrow g(0) = bg^2(0) \Leftrightarrow g(0)(b(g(0)-1) = 0$ and since $g(0) \neq \frac{1}{b}$, it results $g(0) = 0$.
For any $x, y \in \mathbb{R}^*$, we have:

$$g(g(x)f(y)) + af(x)f(y) = bg(f(x))g(f(y)), \text{ respectively}$$

$$g(g(x)f(y)) + af(y)f(x) = bg(f(y))g(f(x)), \text{ so } g(g(x)f(y)) = g(g(y)f(x))$$

and taking account that g is injective we deduce that

$$g(x)f(y) = g(y)f(x), \forall x, y \in \mathbb{R}^* \Leftrightarrow \frac{g(x)}{f(x)} = \frac{g(y)}{f(y)}, \forall x, y \in \mathbb{R}.$$

From the last relation, for $y = 1$, we deduce $\frac{g(x)}{f(x)} = \frac{g(1)}{f(1)} = c \in \mathbb{R}^* \Rightarrow$
$g(x) = cf(x), \forall x \in \mathbb{R}^*$; then from the statement we obtain

$$g(cf(x)f(y)) + af(x)f(y) = bc^2 f(0)f(y), \forall x, y \in \mathbb{R}^* \Rightarrow a = (b-1)c^2 \Leftrightarrow$$

$$\Leftrightarrow c^2 = \frac{a}{b-1} \Rightarrow c = \pm\sqrt{\frac{a}{b-1}}.$$

We get the functions:

$$g_1, g_2 : \mathbb{R} \to \mathbb{R}, g_1(x) = \sqrt{\frac{a}{b-1}} \cdot f(x), g_2(x) = -\sqrt{\frac{a}{b-1}} \cdot f(x)$$

□

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