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If $0 < x < \frac{\pi}{2}$ then:

$$\frac{8\sin^6 x}{1+\cot x}+\frac{8\cos^6 x}{1+\tan x}\geq \sin^3(2x)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco, Solution 2 by Alex Szoros-Romania, Solution 3 by Hafiz Iqbal-Indonesia, Solution 4 by Samar Das-India, Solution 5 by Fayssal Abdelli-Bejaia-Algerie, Solution 6 by Tran Hong-Dong Thap-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$x\in\left(0,\frac{\pi}{2}\right)\Rightarrow \cos x,\sin x>0$$

$$\frac{8\sin^6 x}{1+\cot x}+\frac{8\cos^6 x}{1+\tan x}=\frac{8\sin^7 x}{\sin x+\cos x}+\frac{8\cos^7 x}{\cos x+\sin x}\overset{Holder}{\geq}\frac{8(\sin x+\cos x)^7}{2^6(\sin x+\cos x)}=$$

$$=\frac{1}{8}(\sin x+\cos x)^6\overset{AM-GM}{\geq}\frac{1}{8}\left(2\sqrt{\sin x\cos x}\right)^6=(2\sin x\cos x)^3=\sin^3(2x)$$

Solution 2 by Alex Szoros-Romania

$$\begin{cases}\sin x=a\in(0,1)\\ \cos x=b\in(0,1)\end{cases}\Rightarrow\begin{cases}a^2+b^2=1\\ \sin 2x=ab\end{cases}$$

$$\frac{\sin^6 x}{1+\cot x}=\frac{a^6}{1+\frac{b}{a}}=\frac{a^7}{a+b};\frac{\cos^6 x}{1+\tan x}=\frac{b^7}{a+b}$$

$$\frac{8a^7}{a+b}+\frac{8b^7}{a+b}\ge 8a^3b^3 \Leftrightarrow \frac{a^7+b^7}{a+b}\ge a^3b^3 \Leftrightarrow$$

$$(a^6+b^6)-ab(a^4+b^4)+a^2b^2(a^2+b^2)\ge 2a^3b^3 \Leftrightarrow$$

$$(a^2+b^2)(a^4-a^2b^2+b^4)-ab(a^4+b^4)+a^2b^2\ge 2a^3b^3 \Leftrightarrow$$

$$a^4+b^4-ab(a^4+b^4)\ge 2a^3b^3 \Leftrightarrow$$

$$1-2a^2b^2-ab(1-2a^2b^2)\ge 2a^3b^3 \Leftrightarrow$$

$$1-2a^2b^2-ab+2a^3b^3\ge 2a^3b^3 \Leftrightarrow 2a^2b^2+ab\le 1 \Leftrightarrow$$

$4a^2b^2+2ab\le 2 \Leftrightarrow sin^2(2x)+sin(2x)\le 2$, which is true, because

$sin(2x)\le 1, \forall x\in\mathbb{R}$

Solution 3 by Hafiz Iqbal-Indonesia

$$\frac{8sin^6x}{1+cotx}+\frac{8cos^6x}{1+tanx}=\frac{8sin^6x}{1+\frac{cosx}{sinx}}+\frac{8cos^6x}{1+\frac{sinx}{cosx}}=$$

$$=\frac{8sin^7x}{sinx+cosx}+\frac{8cos^7x}{sinx+cosx}\ge\frac{8(sinx+cosx)^7}{2^6(sinx+cosx)}=$$

$$=\frac{1}{8}(sinx+cosx)^6\overset{AM-GM}{\ge}\frac{1}{8}\left(\sqrt{4sinxcosx}\right)^6=(2sinxcosx)^3=sin^3(2x)$$

Solution 4 by Samar Das-India

$$\frac{8sin^6x}{1+cotx}+\frac{8cos^6x}{1+tanx}=8\left(\frac{sin^7x}{sinx+cosx}+\frac{cos^7x}{sinx+cosx}\right)=$$

$$=\frac{16}{sinx+cosx}\left(\frac{sin^7x+cos^7x}{2}\right)\ge\frac{16}{sinx+cosx}\left(\frac{sinx+cosx}{2}\right)^7=$$

$$=8\left(\frac{sinx+cosx}{2}\right)^6\overset{AM-GM}{\ge}8\left(\sqrt{sinxcosx}\right)^6=8sin^3xcos^3x=sin^3(2x)$$

Solution 5 by Fayssal Abdelli-Bejaia-Algerie

Let suppose that: $\frac{8sin^6x}{1+cotx}+\frac{8cos^6x}{1+tanx}<sin^3(2x);\ \ (A)$

$$\Leftrightarrow\frac{8sin^6x}{\frac{sinx+cosx}{sinx}}+\frac{8cos^6x}{\frac{cosx+sinx}{cosx}}<2^3sin^3x\cdot cos^3x\Leftrightarrow$$

$$\frac{sin^7x}{sinx+cosx}+\frac{cos^7x}{sinx+cosx}<sin^3x+cos^3x\Leftrightarrow$$

$$sin^7x + cos^7x < (sinx + cosx)sin^3x \cdot cos^3x \Leftrightarrow$$

$$sin^7x + cos^7x < sin^4xcos^3x + sin^3xcos^4x \Leftrightarrow$$

$$(sin^3x - cos^3x)(sin^4x - cos^4x) < 0 \Leftrightarrow$$

$$(sinx - cosx)(sin^2x + cos^2x + sinxcosx)(sin^2x - cos^2x) < 0 \Leftrightarrow$$

$(sinx - cosx)^2(sinx + cosx)(1 + sinxcosx) < 0$ contradiction, because

$$\begin{cases}(sinx - cosx)^2 > 0\\ sinx + +cosx > 0\\ 1 + sinxcosx > 0\end{cases}; \forall x \in \left(0, \frac{\pi}{2}\right)$$

Solution 6 by Tran Hong-Dong Thap-Vietnam

Because: $0 < x < \frac{\pi}{2} \Rightarrow 0 < sinx; cosx < 1 \Rightarrow 0 < sin(2x) < 1$

$$\frac{8sin^6x}{1 + cotx} + \frac{8cos^6x}{1 + tanx} = \frac{8(sin^7x + cos^7x)}{sinx + cosx} =$$

$$= \frac{8\left(\frac{sin^8x}{sinx} + \frac{cos^8x}{cosx}\right)}{sinx + cosx} = \frac{8\left(\frac{(sin^4x)^2}{sinx} + \frac{(cos^4x)^2}{cosx}\right)}{sinx + cosx} \overset{BCS}{\geq}$$

$$\overset{BCS}{\geq} \frac{8\left(\frac{1}{2}(sin^2x + cos^2x)^2\right)^2}{(sinx + cosx)^2} = \frac{2}{1 + sin(2x)} \overset{(1)}{\geq} sin^3(2x)$$

$$(1) \Leftrightarrow \frac{2}{1 + t} \geq t^3 (\because t = sin(2x) \Rightarrow t \in (0, 1))$$

$\Leftrightarrow t^4 + t^3 - 2 \leq 0 \Leftrightarrow (t - 1)(t^3 + 2t^2 + 2t + 2) \leq 0$, which is true, because:

$$0 < t \leq 2 \Rightarrow t - 1 \leq 0, t^3 + 2t^2 + 2t + 2 > 0.$$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solutions.