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ROMANIAN MATHEMATICAL MAGAZINE

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If $x, y > 0$ then:

$$\frac{2^x(1+2^{x+2y})}{1+2^x}+\frac{2^y(1+2^{2x+y})}{1+2^y}+\frac{2^{x+y}(2^x+2^y)}{1+2^{x+y}} \geq 3\cdot 2^{x+y}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand, Solution 2 by Samar Das-India, Solution 3 by Tran Hong-Dong Thap-Vietnam

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{2^x(1+2^{x+2y})}{1+2^x}+\frac{2^y(1+2^{2x+y})}{1+2^y}+\frac{2^{x+y}(2^x+2^y)}{1+2^{x+y}} \geq 3\cdot 2^{x+y} \Leftrightarrow$$

$$\frac{(1+2^{x+2y})}{2^y(1+2^x)}+\frac{(1+2^{x+2y})}{2^x(1+2^y)}+\frac{2^x+2^y}{1+2^{x+y}} \geq 3$$

$$3\sqrt[3]{\frac{(1+2^{x+2y})(1+2^{2x+y})(2^x+2^y)}{2^{x+y}(1+2^x)(1+2^y)(1+2^{x+y})}} \geq 3$$

$$(1+2^{x+2y})(1+2^{2x+y})(2^x+2^y) \geq 2^{x+y}(1+2^x)(1+2^y)(1+2^{x+y})$$

$$2^x+2^{3x+y}+2^{2x+2y}+2^{4x+3y}+2^y+2^{2x+2y}+2^{x+3y}+2^{3x+4y} \geq$$

$\geq 2^{x+y}+2^{x+2y}+2^{2x+y}+2^{2x+2y}+2^{2x+3y}+2^{3x+2y}+2^{3x+3y}$, **which is true, because**

$$2^x+2^{3x+4y} \geq 2^{x+y}+2^{3x+3y};\ 2^y+2^{3x+y} \geq 2^{x+y}+2^{2x+y} |: 2^x, 2^y \geq 2$$

$$2^{(4x+3y)}+2^{x+3y}+2^{x^2+y}\geq 2^{x+2y}+2^{2x+3y}+2^{3x+2y}$$

$$2^{3x+2y}+2^{2y}+1\geq 2^{y}+2^{2x+3y}+2^{3x+2y}$$

$$2^{3x+2y}+2^{y}\geq 2^{x+2y}+2^{2x+y}$$

$$2^{3x+y}+1\geq 2^{x+y}+2^{2x}, \text{ true.}$$

Solution 2 by Samar Das-India

$$\frac{2^x(1+2^{x+2y})}{1+2^x}+\frac{2^y(1+2^{2x+y})}{1+2^y}+\frac{2^{x+y}(2^x+2^y)}{1+2^{x+y}}\overset{AM-GM}{\geq}$$

$$\geq 3\cdot\sqrt[3]{\frac{(1+2^{x+2y})(1+2^{2x+y})(2^x+2^y)}{2^{x+y}(1+2^x)(1+2^y)(1+2^{x+y})}}\geq 3\cdot\sqrt[3]{\frac{2^{2x+2y}\left(2\sqrt{2^{x+2y}}\right)\left(2\sqrt{2^{2x+y}}\right)\left(2\sqrt{2^x2^y}\right)}{(1+2^x)(1+2^y)(1+2^{x+y})}}$$

$$=3\cdot 2\cdot\sqrt[3]{\frac{2^{2x+2y}\cdot 2^{\frac{x+2y+2x+y+x+y}{2}}}{(1+2^x)(1+2^y)(1+2^{x+y})}}=6\cdot\frac{\sqrt[3]{2^{4(x+y)}}}{\sqrt[3]{(1+2^x)(1+2^y)(1+2^{x+y})}}=$$

$$=\frac{6\cdot 2^{x+y}}{2^{\frac{x+y}{3}}\cdot\sqrt[3]{(1+2^x)(1+2^y)(1+2^{x+y})}}=3\cdot 2^{x+y}$$

Solution 3 by Tran Hong-Dong Thap-Vietnam

Let $a=2^x; b=2^y; c=2^z; (x,y\in\mathbb{N}\Rightarrow a,b\geq 1)$. Inequality becomes as:

$$\frac{a(1+ab^2)}{1+a}+\frac{b(1+a^2b)}{1+b}+\frac{ab(a+b)}{1+ab}\geq 3ab; (1)$$

If $a=1, b\geq 2$ then $\frac{1+b^2}{2}+b+b\geq\frac{2b}{b}+b+b=3b\Rightarrow (1)$ is true.

If $b=1, a\geq 1$ then: $\frac{1+a^2}{1}+a+a\geq\frac{2a}{2}+a+a=3a\Rightarrow (1)$ is true.

If $a\geq 2, b\geq 2\Rightarrow 1+a\leq\frac{a}{2}+a=\frac{3a}{2}, 1+b\leq\frac{b}{2}+b=\frac{3b}{2}, 1+ab\leq\frac{ab}{4}+ab=\frac{5ab}{4}\Rightarrow$

$$\frac{a(1+ab^2)}{1+a}+\frac{b(1+a^2b)}{1+b}+\frac{ab(a+b)}{1+ab}\geq\frac{2(1+ab^2)}{3}+\frac{2(1+a^2b)}{3}+\frac{4(a+b)}{5}=$$

$$=\frac{5\big(4+2ab(a+b)\big)+12(a+b)}{15}=\frac{20+2(a+b)(5ab+6)}{15}\overset{AM-GM}{\geq}$$

$$\geq\frac{20+4\sqrt{ab}(5ab+6)}{15}\overset{t=\sqrt{ab}\geq 2}{=}\frac{20+4t(5t^2+6)}{15}\overset{(2)}{>}3t^3=3ab$$

$$(2)\Leftrightarrow 20t^3+24t+20-45t^2>0\Leftrightarrow t(20t^2-45t+24)+20>0\Leftrightarrow$$

$t\left[20\left(t-\frac{9}{8}\right)^2-\frac{21}{16}\right]+20>0$, **which is true because:** $t\geq 2\Rightarrow$

$\left[20\left(t-\frac{9}{8}\right)^2-\frac{21}{16}\right]\geq 2\left[20\left(2-\frac{9}{8}\right)^2-\frac{21}{16}\right]\geq 14>0\Rightarrow(2)\Rightarrow(1)$ **is true.**

Equality holds if $a=b=1\Leftrightarrow x=y=0$.

Note by editor:
Many thanks to Florică Anastase-Romania for typed solutions.