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If $x,y,z,t>0$ then:

$$\frac{x}{2y}+\frac{y}{2z}+\frac{z}{2t}+\frac{t}{2x}+\frac{16xyzt}{(x+y)(y+z)(z+t)(t+x)}\geq 3$$

Proposed by Marin Chirciu-Romania

Solution 1 by George Florin Şerban-Romania, Solution 2 by Tran Hong-Dong Thap-Vietnam

Solution 1 by George Florin Şerban-Romania

$$\frac{x}{2y}+\frac{y}{2z}+\frac{z}{2t}+\frac{t}{2x}+\frac{16xyzt}{(x+y)(y+z)(z+t)(t+x)}\geq 3\Leftrightarrow$$

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{t}+\frac{t}{x}+\frac{32xyzt}{(x+y)(y+z)(z+t)(t+x)}\geq 6\Leftrightarrow$$

$$\left(\frac{x}{y}+1\right)+\left(\frac{y}{z}+1\right)+\left(\frac{z}{t}+1\right)+\left(\frac{t}{x}+1\right)+\frac{32xyzt}{(x+y)(y+z)(z+t)(t+x)}\geq 10\Leftrightarrow$$

$\frac{x+y}{x}+\frac{y+z}{z}+\frac{z+t}{t}+\frac{t+x}{x}+\frac{32xyzt}{(x+y)(y+z)(z+t)(t+x)}\geq 10$, which is true from AM-GM:

$$\frac{x+y}{x}+\frac{y+z}{z}+\frac{z+t}{t}+\frac{t+x}{x}+\frac{32xyzt}{(x+y)(y+z)(z+t)(t+x)}\geq$$

$$\geq 5\cdot\sqrt[5]{\frac{x+y}{x}\cdot\frac{y+z}{z}\cdot\frac{z+t}{t}\cdot\frac{t+x}{x}\cdot\frac{32xyzt}{(x+y)(y+z)(z+t)(t+x)}}=5\cdot\sqrt[5]{32}=10$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

Let $u=\frac{x}{2y}, v=\frac{y}{2z}, m=\frac{z}{2t}, n=\frac{t}{2x}\Rightarrow uvmn=\frac{1}{16}\Rightarrow$

$$u+v+m+n\overset{AM-GM}{\geq}4\cdot\sqrt[4]{uvmn}=2$$

Now, inequality becomes as:

$$u+v+m+n+\frac{1}{\left(u+\frac{1}{2}\right)\left(v+\frac{1}{2}\right)\left(m+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)}\geq 3;(*)$$

But: $\left(u+\frac{1}{2}\right)\left(v+\frac{1}{2}\right)\left(m+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)\overset{AM-GM}{\leq}\left(\frac{u+v+m+n}{4}\right)^4$

Let: $X=u+v+m+n+2\geq 4$, we need to prove that:

$X+\frac{1}{\left(\frac{X}{4}\right)^4}\geq 5\Leftrightarrow X^5-5X^4+256\geq 0\Leftrightarrow (X-4)^2(X^3+3X^2+8X+16)\geq 0$, which is

true because $X\geq 4\Rightarrow(*)$ is true.

Equality holds for $X=4\Leftrightarrow u=v=m=n=\frac{1}{2}\Leftrightarrow x=y=z=t$.

Note by editor:
Many thanks to Florică Anastase-Romania for typed solutions.