R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

If $0<\alpha<x,y,z\le\beta$ then:

$$\left(\sqrt[3]{xyz}+\frac{x+y+z}{3}+\sqrt{\frac{x^2+y^2+z^2}{3}}\right)\left(\frac{1}{\sqrt[3]{xyz}}+\frac{3}{x+y+z}+\sqrt{\frac{3}{x^2+y^2+z^2}}\right)\le\frac{9(\alpha+\beta)^2}{4\alpha\beta}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco, Solution 2 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let: $m_g=\sqrt[3]{xyz};m_a=\frac{x+y+z}{3};m_q=\sqrt{\frac{x^2+y^2+z^2}{3}}$**, We know that:** $\alpha\le m_g\le m_a\le m_h\le\beta$

$$(m_g+m_a+m_q)\left(\frac{1}{m_g}+\frac{1}{m_a}+\frac{1}{m_q}\right)=3+\left(\frac{m_g}{m_a}+\frac{m_a}{m_g}\right)+\left(\frac{m_q}{m_g}+\frac{m_g}{m_q}\right)+\left(\frac{m_q}{m_g}+\frac{m_g}{m_q}\right)$$

$$(m_q-m_a)(m_a-m_g)\ge 0\Leftrightarrow m_qm_a+m_am_g\ge m_qm_g+m_a^2\Rightarrow$$

$$1+\frac{m_g}{m_q}\ge\frac{m_g}{m_a}+\frac{m_a}{m_q}\text{ and }\frac{m_q}{m_g}+1\ge\frac{m_q}{m_a}+\frac{m_a}{m_g}$$

$$\left(\frac{m_g}{m_a}+\frac{m_a}{m_g}\right)+\left(\frac{m_q}{m_a}+\frac{m_a}{m_q}\right)\le 2+\left(\frac{m_g}{m_q}+\frac{m_q}{m_g}\right)$$

$$\Rightarrow\left(\sum_{cyc} m_a\right)\left(\sum_{cyc}\frac{1}{m_a}\right)\le 5+2\left(\frac{m_g}{m_q}+\frac{m_q}{m_g}\right)$$

$$m_q,m_g\in[\alpha,\beta]\Rightarrow m=\frac{m_g}{m_q}\in\left[\frac{\alpha}{\beta},\frac{\beta}{\alpha}\right]\Rightarrow$$

$$\left(\frac{\beta}{\alpha}-m\right)\left(m-\frac{\alpha}{\beta}\right)\ge 0\Rightarrow\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)m\ge m^2+1$$

$$\Rightarrow\frac{m_g}{m_q}+\frac{m_q}{m_g}=m+\frac{1}{m}\le\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}$$

$$\Rightarrow\left(\sum_{cyc} m_g\right)\left(\sum_{cyc}\frac{1}{m_g}\right)\le 5+2\cdot\frac{\alpha^2+\beta^2}{\alpha\beta}\overset{(1)}{\le}\frac{9(\alpha+\beta)^2}{4\alpha\beta}$$

$(1)\Leftrightarrow 2\alpha\beta\le\alpha^2+\beta^2\Leftrightarrow(\alpha-\beta)^2\ge 0$, which is true.

Therefore,

$$\left(\sqrt[3]{xyz}+\frac{x+y+z}{3}+\sqrt{\frac{x^2+y^2+z^2}{3}}\right)\left(\frac{1}{\sqrt[3]{xyz}}+\frac{3}{x+y+z}+\sqrt{\frac{3}{x^2+y^2+z^2}}\right)\le$$

$$\le\frac{9(\alpha+\beta)^2}{4\alpha\beta}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$0<\alpha<x,y,z\le\beta; a=\sqrt[3]{xyz}; b=\frac{x+y+z}{3}; c=\sqrt{\frac{x^2+y^2+z^2}{3}}$$

Hence $\alpha\le a,b,c\le\beta$. Since, $\alpha\le a\le\beta\Rightarrow(a-\alpha)(\beta-a)\ge 0\Rightarrow$

$$\frac{\alpha+\beta}{\sqrt{\alpha\beta}}\ge\frac{a}{\sqrt{\alpha\beta}}+\frac{\sqrt{\alpha\beta}}{a};\ \frac{\alpha+\beta}{\sqrt{\alpha\beta}}\ge\frac{b}{\sqrt{\alpha\beta}}+\frac{\sqrt{\alpha\beta}}{b};\ \frac{\alpha+\beta}{\sqrt{\alpha\beta}}\ge\frac{c}{\sqrt{\alpha\beta}}+\frac{\sqrt{\alpha\beta}}{c}$$

$$\left(\sqrt[3]{xyz}+\frac{x+y+z}{3}+\sqrt{\frac{x^2+y^2+z^2}{3}}\right)\left(\frac{1}{\sqrt[3]{xyz}}+\frac{3}{x+y+z}+\sqrt{\frac{3}{x^2+y^2+z^2}}\right)=$$

$$=(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\left(\sum_{cyc}\frac{a}{\sqrt{\alpha\beta}}\right)\left(\sum_{cyc}\frac{\sqrt{\alpha\beta}}{a}\right)\overset{AM-GM}{\le}$$

$$\leq \frac{1}{4}\left[\sum_{cyc}\left(\frac{a}{\sqrt{\alpha\beta}}+\frac{\sqrt{\alpha\beta}}{a}\right)\right]^2=\frac{9(\alpha+\beta)^2}{4\alpha\beta}$$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solutions.