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If $a, b, c > 0$ and $a^3 + b^3 + c^3 = 1, \lambda \geq 0$ then:

$$\frac{a^4}{b + \lambda c} + \frac{b^4}{c + \lambda a} + \frac{c^4}{a + \lambda b} \geq \frac{1}{\lambda + 1}$$

Proposed by Marin Chirciu-Romania

Solution 1 by George Florin Şerban-Romania, Solution 2 by Ruxandra Daniela Tonilă-Romania

Solution 1 by George Florin Şerban-Romania

$$a^3 + b^3 + b^3 \geq 3\sqrt[3]{a^3b^3b^3} = 3ab^2; (by\ AM - GM)$$

$$b^3 + c^3 + c^3 \geq 3\sqrt[3]{b^3c^3c^3} = 3bc^2$$

$$c^3 + a^3 + a^3 \geq 3\sqrt[3]{c^3a^3a^3} = 3ca^2$$

$$3\sum_{cyc} a^3 \geq 3(ab^2 + bc^2 + ca^2) \Rightarrow \sum_{cyc} a^3 \geq ab^2 + bc^2 + ca^2$$

$$a^3 + c^3 + c^3 \geq 3\sqrt[3]{a^3c^3c^3} = 3ac^2; (by\ AM - GM)$$

$$b^3 + a^3 + a^3 \geq 3\sqrt[3]{a^3b^3b^3} = 3ba^2$$

$$c^3 + b^3 + b^3 \geq 3\sqrt[3]{c^3b^3b^3} = 3cb^2 \Rightarrow 3\sum_{cyc} a^3 \geq 3(ac^2 + a^2b + b^2c)$$

$$\sum_{cyc} a^3 \ge ac^2 + cb^2 + ba^2 \Rightarrow \sum_{cyc} \frac{a^4}{b + \lambda c} = \sum_{cyc} \frac{a^6}{a^2 b + \lambda a^2 c} = \sum_{cyc} \frac{(a^3)^2}{a^2 b + \lambda a^2 c} \ge$$

$$\overset{Bergstrom}{\ge} \frac{(a^3 + b^3 + c^3)^2}{(a^2 b + b^2 c + c^2 a) + \lambda(a^2 c + b^2 a + c^2 b)} \ge \frac{1}{\sum a^3 + \lambda \sum a^3} = \frac{1}{1 + \lambda}$$

Therefore,

$$\frac{a^4}{b + \lambda c} + \frac{b^4}{c + \lambda a} + \frac{c^4}{a + \lambda b} \ge \frac{1}{\lambda + 1}$$

Solution 2 by Ruxandra Daniela Tonilă-Romania

$$\frac{a^4}{b + \lambda c} + \frac{b^4}{c + \lambda a} + \frac{c^4}{a + \lambda b} = \frac{\frac{a^4}{b + \lambda c} + \frac{b^4}{c + \lambda a} + \frac{c^4}{a + \lambda b}}{1} =$$

$$= \frac{\frac{a^4}{b + \lambda c} + \frac{b^4}{c + \lambda a} + \frac{c^4}{a + \lambda b}}{a^3 + b^3 + c^3} = \frac{a^3 \frac{a}{b + \lambda c} + b^3 \frac{b}{c + \lambda a} + c^3 \frac{c}{a + \lambda b}}{a^3 + b^3 + c^3} \overset{AM-GM}{\ge}$$

$$\ge \frac{a^3 + b^3 + c^3}{a^3 \left(\frac{b + \lambda c}{a}\right) + b^3 \left(\frac{c + \lambda a}{b}\right) + c^3 \left(\frac{a + \lambda b}{c}\right)} \Leftrightarrow$$

$$\frac{a^4}{b + \lambda c} + \frac{b^4}{c + \lambda a} + \frac{c^4}{a + \lambda b} \ge \frac{1}{a^2(b + \lambda c) + b^2(c + \lambda a) + c^2(a + \lambda b)} \Leftrightarrow$$

$$\frac{a^4}{b + \lambda c} + \frac{b^4}{c + \lambda a} + \frac{c^4}{a + \lambda b} \ge \frac{1}{(a^2 b + b^2 c + c^2 a) + \lambda(a^2 c + b^2 a + c^2 b)}; (1)$$

$$a^3 + b^3 + c^3 \ge a^2 b + b^2 c + c^2 a \Rightarrow 1 \ge a^2 b + b^2 c + c^2 a; (2)$$

$$a^3 + b^3 + c^3 \ge b^2 a + bc^2 + a^2 c \Rightarrow 1 \ge ab^2 + bc^2 + ca^2 | \cdot \lambda \Rightarrow$$

$$\lambda \ge (ab^2 + bc^2 + ca^2)\lambda; (3)$$

From (1),(2),(3) it follows that:

$$\frac{1}{(a^2 b + b^2 c + c^2 a) + \lambda(a^2 c + b^2 a + c^2 b)} \le \frac{1}{\lambda + 1}; (4)$$

From (1),(4) we get:

$$\frac{a^4}{b + \lambda c} + \frac{b^4}{c + \lambda a} + \frac{c^4}{a + \lambda b} \ge \frac{1}{a^2(b + \lambda c) + b^2(c + \lambda a) + c^2(a + \lambda b)} \ge \frac{1}{\lambda + 1}$$

Therefore,

$$\frac{a^4}{b + \lambda c} + \frac{b^4}{c + \lambda a} + \frac{c^4}{a + \lambda b} \ge \frac{1}{\lambda + 1}$$

Note by editor:

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