Solution attempt by Long Huynh Huu (@erugli) for Dan Sitaru's inequality which was posted on Twitter by Nassim Taleb (@nntaleb) [1].
October 18$^{\text{th}}$ 2020

1 Schur convexity

A classical application of Schur convexity goes as follows: If a sequence of positive real numbers x, y, z majorises another such sequence a, b, c, then

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \tag{1}$$

In this document I want to prove an extension of this result to integrals.

2 Extension to integrals

Theorem 1

Let $F : I \to \mathbb{R}$ be a convex Lipschitz function on an open interval $I \subset \mathbb{R}$.
Let $u, v : [a, b] \to I$ be monotonically increasing Lipschitz functions on the interval $[a, b]$, such that

$$\int_a^b u(t)\, dt = \int_a^b v(t)\, dt \tag{2}$$

$$\int_a^x u(t)\, dt \leq \int_a^x v(t)\, dt \quad (x \in [a, b]) \tag{3}$$

Then

$$\int_a^b F(u(t))\; dt \geq \int_a^b F(v(t))\; dt \tag{4}$$

Proof. Let $n > 0$ be a natural number. Partition $(a, b]$ into n intervals $I_i = a + (b - a) \cdot \left(\frac{i-1}{n}, \frac{i}{n}\right]$ with $i \in [n]$. We get two increasing sequences

$$u_i = \int_{I_i} u(t)\, dt \quad (i \in [n])$$

$$v_i = \int_{I_i} v(t)\, dt \quad (i \in [n])$$

The conditions of Theorem 1 imply that the u_i majorise the v_i, so by Schur Majorisation Inequality (Problem 13.4 of [2]).

$$\sum_{i=1}^{n} F(nu_i) \geq \sum_{i=1}^{n} F(nv_i)$$

$$\iff \sum_{i=1}^{n} F\left(n \int_{I_i} u(t)\, dt\right) \geq \sum_{i=1}^{n} F\left(n \int_{I_i} v(t)\, dt\right)$$

The theorem follows from proving the following limit for u (and the analogous version for v):

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F\left(n \int_{I_i} u(t)\, dt\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} n \int_{I_i} F(u(t))\, dt = \int_a^b F(u(t))\, dt$$

Let $m(I_i) = a + (b-a)\frac{2i-1}{2n}$ be the midpoint of I_i. Let $L > 0$ be the Lipschitz constant for u and let K be the Lipschitz constant for F.

$$\left| n \int_{I_i} u(t)\, dt - u(m(I_i)) \right| \leq n \int_{I_i} |u(t) - u(m(I_i))|\, dt \leq \frac{L}{n} \tag{5}$$

$$\left| F\left(n \int_{I_i} u(t)\, dt\right) - F(u(m(I_i))) \right| \leq K \left| n \int_{I_i} u(t)\, dt - u(m(I_i)) \right| \leq \frac{KL}{n} \tag{6}$$

$$\left| n \int_{I_i} F(u(t))\, dt - F(u(m(I_i))) \right| \leq n \int_{I_i} |F(u(t)) - F(u(m(I_i)))|\, dt \leq \frac{KL}{n} \tag{7}$$

Inequalities (5) and (7) are due to Lipschitz continuity of u, and $F \circ u$ respectively. Inequality (5) implies (6). Inequalities (6) and (7) together imply

$$\begin{aligned}
&\left| n \int\limits_{I_i} F(u(t))\,dt - nF\left(\int\limits_{I_i} u(t)\,dt \right) \right| \\
\leq &\left| n \int\limits_{I_i} F(u(t))\,dt - F(u(m(I_i))) \right| + \left| F(u(m(I_i))) - nF\left(\int\limits_{I_i} u(t)\,dt \right) \right| \\
\leq &\frac{2KL}{n}
\end{aligned} \tag{8}$$

Therefore

$$\frac{1}{n} \sum_{i=1}^{n} F\left(n \int\limits_{I_i} u(t)\,dt \right) = \frac{1}{n} \sum_{i=1}^{n} n \int\limits_{I_i} F(u(t))\,dt + \mathcal{O}\left(\frac{2KL}{n} \right) = \int\limits_a^b F(u(t))\,dt + \mathcal{O}\left(\frac{2KL}{n} \right)$$

□

3 Simplifying the condition

Corollary 1

Let $F : I \to \mathbb{R}$ be a convex Lipschitz function on an open interval $I \subset \mathbb{R}_+$.
Let $f, g : [a, b] \to I$ be monotonically increasing Lipschitz functions on the interval $[a, b]$, such that

$$f(x)g(x) > 0 \quad (a < x < b) \tag{9}$$

$$\frac{\int\limits_a^x f(s)\,ds}{\int\limits_a^x g(s)\,ds} \text{ is non-decreasing with respect to } x \in (a, b) \tag{10}$$

Then

$$\int\limits_a^b F\left(\frac{f(t)}{\int\limits_a^b f(s)\,ds} \right) dt \geq \int\limits_a^b F\left(\frac{g(t)}{\int\limits_a^b g(s)\,ds} \right) dt \tag{11}$$

Proof. Set $u(x) = \frac{f(x)}{\int_a^b f(s)\,ds}$ and $v(x) = \frac{f(x)}{\int_a^b g(s)\,ds}$. By construction u and v satisfy equation (2).
The second condition (3) requires for $a < x < b$:

$$\frac{\int_a^x f(s)\,ds}{\int_a^b f(s)\,ds} \le \frac{\int_a^x g(s)\,ds}{\int_a^b g(s)\,ds}$$

$$\iff \frac{\int_a^x f(s)\,ds}{\int_a^x g(s)\,ds} \le \frac{\int_a^b f(s)\,ds}{\int_a^b g(s)\,ds}$$

This inequality holds because the left-hand term is non-decreasing in x, while equality holds for $x = b$. Therefore Theorem 1 applies. □

4 Application to Dan Sitaru's inequality

Dan Sitaru observed that

$$\int_a^b (\log x)^{\log x}\,dx \cdot \int_a^b (\log x)^{-\log x}\,dx \ge (b^2 - a^2)\log\sqrt{\frac{b}{a}} \qquad (e \le a \le b) \tag{12}$$

which is equivalent to saying

$$\int_a^b \frac{\int_a^b (\log x)^{\log x}\,dx}{(\log x)^{\log x}}\,dx \ge \int_a^b \frac{\int_a^b x\,dx}{x}\,dx \tag{13}$$

This follows from Corollary 1 with $F(x) = \frac{1}{x}$, $f(x) = \log(x)^{\log(x)}$, and $g(x) = x$. Note that f and g are increasing functions on (e, ∞). Because f and g are strictly positive for $x \ge e$, the positivity condition (9) is satisfied.
We will show the monotonicity condition (10) by taking the derivative.

$$\frac{\partial}{\partial x}\frac{\int_a^x \log(s)^{\log(s)}\,ds}{\int_a^x s\,ds} \geq 0$$

$$\Longleftrightarrow \quad \frac{\log(x)^{\log(x)}\frac{x^2-a^2}{2} - x\int_a^x \log(s)^{\log(s)}\,ds}{\left(\int_a^x s\,ds\right)^2} \geq 0$$

$$\Longleftrightarrow \quad \log(x)^{\log(x)}\frac{x^2-a^2}{2} \geq x\int_a^x \log(s)^{\log(s)}\,ds \tag{14}$$

Because $\log(s)^{\log(s)}$ is convex on (e,∞), the mean on $[a,x]$ is bounded by the mean of the values at the endpoints.

$$\frac{1}{x-a}\int_a^x \log(s)^{\log(s)} \leq \frac{\log(x)^{\log(x)} + \log(a)^{\log(a)}}{2} \tag{15}$$

Due to monotonicity of $\frac{log(x)^{\log(x)}}{x}$ on (e,∞), we further get

$$\frac{\log(x)^{\log(x)} + \frac{a}{a}\log(a)^{\log(a)}}{2} \leq \frac{\log(x)^{\log(x)} + \frac{a}{x}\log(x)^{\log(x)}}{2} = \frac{x+a}{2x}\log(x)^{\log(x)} \tag{16}$$

$$\overset{(15),(16)}{\Longrightarrow} x\int_a^x \log(s)^{\log(s)} \leq \frac{x+a}{2}(x-a)\log(x)^{\log(x)}$$

Hence we have proven inequality (14) to hold.

5 References

[1] N. Taleb, *To kill time today. There must be a trick.*, https://twitter.com/nntaleb/status/1316693195506548736 (2020).

[2] J.M. Steele, *The Cauchy-Schwarz master class: an introduction to the art of mathematical inequalities* (Cambridge University Press, 2004).