AMAZING LINEAR RECURRENCES OF POSITIVE REAL SEQUENCES

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ABSTRACT. In this paper we present some certain results on positive real sequences.

Theorem 1. If $(x_n)_{n\geq 0}, (y_n)_{n\geq 0}$ are positive real sequences with $x_{n+2} = x_{n+1} + x_n, y_{n+2} = -y_{n+1} + y_n; \forall n \in \mathbb{N}$, then

$$\sum_{k=0}^{n} \binom{n}{k} (x_k + (-1)^k y_k) = x_{2n} + y_{2n}, \forall n \in \mathbb{N}$$

Proof. We have that: $r^2 - r - 1 = 0, t^2 + t - 1 = 0:$

$$r_1 = \frac{1+\sqrt{5}}{2} = \alpha, r_2 = \frac{1-\sqrt{5}}{2} = \beta, t_1 = \frac{-1+\sqrt{5}}{2} = -\beta, t_2 = \frac{-1-\sqrt{5}}{2} = -\alpha$$

$$\text{So, } x_n = A\alpha^n + B\beta^n, y_n = (C\alpha^n + D\beta^n)(-1)^n, \forall n \in \mathbb{N}$$

Hence:

$$\sum_{k=0}^{n} \binom{n}{k} (x_k + (-1)^k y_k) = \sum_{k=0}^{n} \binom{n}{k} (A\alpha^k + B\beta^k + C\alpha^k + D\beta^k) =$$

$$= (A+C)\sum_{k=0}^{n} \binom{n}{k} \alpha^k + (B+D)\sum_{k=0}^{n} \binom{n}{k} \beta^k = (A+C)(\alpha+1)^n + (B+D)(\beta+1)^n =$$

$$= (A+C)a^{2n} + (B+D)\beta^{2n} = A\alpha^{2n} + B\beta^{2n} + (C\alpha^{2n} + D\beta^{2n})(-1)^{2n} = x_{2n} + y_{2n}$$

□

Theorem 2. If $a, b \in \mathbb{R}^*$ and $(x_n)_{n\geq 0}, (y_n)_{n\geq 0}$ are real sequences with $x_{n+2} = \alpha x_{n+1} + bx_n, y_{n+2} = -ay_{n+1} + by_n, \forall n \in \mathbb{N}$, then:

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} (x_k + (-1)^k y_k) = x_{2n} + y_{2n}, \forall n \in \mathbb{N}$$

Proof. Case 1. $a^2 + 4b \in \mathbb{R}^*$. So:

$$r^2 - ar - b = 0 \text{ and } t^2 + at - b = 0$$

with $r_1, r_2 \in \mathbb{C}^*, r_1 \neq r_2$, respectively $t_1 = -r_2, t_2 = -r_1$. Hence:

$$x_n = Ar_1^n + Br_2^n, y_n = (-1)^n(Cr_1^n + Dr_2^n), \forall n \in \mathbb{N},$$

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} (x_k + (-1)^k y_k) = A\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} r_1^k + B\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} r_2^k +$$

$$+C\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}(-1)^{2k}r_1^k+D\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}(-1)^{2k}r_2^k=$$

$$=A(ar_1+b)^n+B(ar_2+b)^n+C(ar_1+b)^n+D(ar_2+b)^n=$$

$$=(A+C)r_1^{2n}+(B+D)r_2^{2n}=x_{2n}+Cr_1^n+Dr_2^n=x_{2n}+(-1)^{2n}(Cr_1^{2n}+Dr_2^{2n})=$$

$$=x_{2n}+y_{2n},\forall n\in\mathbb{N}.$$

Case 2. $a^2+4b=0\Rightarrow b=-\left(\frac{a}{2}\right)^2$, then $r_1=r_2=\frac{a}{2}, t_1=t_2=-\frac{a}{2}$.

$$x_n=\frac{(A+Bn)a^n}{2^n}, y_n=\frac{(C+Dn)a^n(-1)^n}{2^n},\forall n\in\mathbb{N},$$

$$x_{2n}=\frac{(A+2Bn)a^{2n}}{2^{2n}}, y_{2n}=\frac{(C+2Dn)a^{2n}}{2^{2n}},\forall n\in\mathbb{N}$$

Therefore:

$$\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}(x_k+(-1)^ky_k)=\sum_{k=0}^{n}\binom{n}{k}a^k(-1)^{n-k}\cdot\frac{a^{2n-2k}}{2^{2n-2k}}(x_k+(-1)^ky_k)=$$

$$=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\cdot\frac{a^{2n-k}}{a^{2n-2k}}\cdot\frac{(A+Bk)a^k}{2^k}+\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\cdot\frac{a^{2n-k}}{a^{2n-2k}}\cdot\frac{(C+Dk)a^k}{2^k}=$$

$$=(A+C)\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\cdot\frac{a^{2n}}{a^{2n-k}}+(B+D)\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\cdot\frac{ka^{2n}}{2^{2n-k}}=$$

$$=\left(\frac{a}{2}\right)^{2n}(A+C)\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}2^k+\left(\frac{a}{2}\right)^{2n}(B+D)\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k\cdot 2^k=$$

$$=(A+C)\cdot\left(\frac{a}{2}\right)^{2n}(2-1)^n+(B+D)\cdot\left(\frac{a}{2}\right)^{2n}\sum_{k=1}^{n}\frac{n!}{k!(n-k)!}\cdot k\cdot 2^k(-1)^{n-k}=$$

$$=(A+C)\cdot\left(\frac{a}{2}\right)^{2n}+(B+D)\cdot\left(\frac{a}{2}\right)^{2n}\cdot\sum_{k=1}^{n}\frac{(n-1)!}{(k-1)!(n-k)!}\cdot 2^{k-1}\cdot(-1)^{n-k}=$$

$$=(A+C)\cdot\left(\frac{a}{2}\right)^{2n}+2n(B+D)\cdot\left(\frac{a}{2}\right)^{2n}\sum_{j=0}^{n-1}\binom{n-1}{j}(2)^{j-1}(-1)^{n-1-j}=$$

$$=(A+C)\cdot\left(\frac{a}{2}\right)^{2n}+2n(B+D)\cdot\left(\frac{a}{2}\right)^{2n}(2-1)^{n-1}=$$

$$=(A+C)\cdot\left(\frac{a}{2}\right)^{2n}+2n(B+D)\cdot\left(\frac{a}{2}\right)^{2n}=\frac{(A+2nB)a^{2n}}{2^{2n}}+\frac{(C+2nD)a^{2n}}{2^{2n}}=$$

$$=x_{2n}+y_{2n},\forall n\in\mathbb{N},\text{ and we are done.}$$

□

Theorem 3. If $a,b\in\mathbb{R}^*$, and $(x_n)_{n\geq 0}$ is real sequence with $x_{n+2}=ax_{n+1}+bx_n,\forall n\in\mathbb{N}$, then:

$$\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}x_k=x_{2n},\forall n\in\mathbb{N}.$$

Proof. Case 1. $r_1, r_2 \in \mathbb{C}^*, r_1 \neq r_2$. Then:

$$x_n = Ar_1^n + Br_2^n, \forall n \in \mathbb{N} \text{ with } A, B \in \mathbb{C}, \text{ so:}$$

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} x_k = A \cdot \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} r_1^k + B \cdot \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} r_2^k =$$

$$= A(ar_1 + b)^n + B(ar_2 + b)^n = Ar_1^{2n} + Br_2^{2n} = x_{2n}, \forall n \in \mathbb{N}.$$

Case 2. $r_1 = r_2 = -\left(\frac{a}{2}\right)^2$. Then:

$$x_n = \frac{(A + Bn)a^n}{2^n}, x_{2n} = \frac{(A + 2Bn)a^{2n}}{2^{2n}}, \text{ so:}$$

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} x_k = \sum_{k=0}^{n} \binom{n}{k} \frac{a^k b^{n-k} a^{2n-2k}}{2^{2n-2k}} x_k = A \cdot \sum_{k=0}^{n} \binom{n}{k} \frac{a^{2n-k}(-1)^{n-k}}{2^{2n-2k}} \cdot \frac{a^k}{2^k} +$$

$$+B \cdot \sum_{k=0}^{n} \binom{n}{k} \frac{a^{2n-k}(-1)^{n-k}}{2^{2n-2k}} \cdot k \cdot \frac{a^k}{2^k} = A \cdot \sum_{k=0}^{n} \binom{n}{k} \frac{a^{2n}(-1)^{n-k}}{2^{2n-k}} +$$

$$+B \cdot \sum_{k=0}^{n} \binom{n}{k} \frac{a^{2n}(-1)^{n-k}}{2^{2n-k}} \cdot k =$$

$$= A \cdot \left(\frac{a}{2}\right)^{2n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} 2^k + B \cdot \left(\frac{a}{2}\right)^{2n} \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} k \cdot 2^k =$$

$$= A \cdot \left(\frac{a}{2}\right)^{2n} + B \cdot \left(\frac{a}{2}\right)^{2n} \cdot \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} \cdot \frac{k}{n} \cdot 2^{k-1} =$$

$$= A \cdot \left(\frac{a}{2}\right)^{2n} + B \cdot \left(\frac{a}{2}\right)^{2n} \cdot 2n \cdot \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} (-1)^{n-k} 2^{k-1} =$$

$$= A \cdot \left(\frac{a}{2}\right)^{2n} + B \cdot \left(\frac{a}{2}\right)^{2n} \cdot \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^{n-j-1} \cdot 2^j =$$

$$= A \cdot \left(\frac{a}{2}\right)^{2n} + B \cdot \left(\frac{a}{2}\right)^{2n} \cdot 2n \cdot (2-1)^{n-1} = (A + 2nB)\left(\frac{a}{2}\right)^{2n} = x_{2n}, \forall n \in \mathbb{N}.$$

□

Theorem 4. If $a, b \in \mathbb{R}^*, 4a^3 + 27b \in \mathbb{R}^*$, and $(x_n)_{n \geq 0}$ is a real sequence such that $x_{n+3} = ax_{n+2} + bx_n, \forall n \in \mathbb{N}$, then:

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} x_{2k} = x_{3n}, \forall n \in \mathbb{N}.$$

Proof. $r^3 - ar^2 - b = 0$, with distinct roots because $4a^3 + 27b \in \mathbb{R}^*$. Let $u, v, w \in \mathbb{C}$ be these roots so $x_n = Au^n + Bv^n + Cw^n, \forall n \in \mathbb{N}$. Hence,

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} x_{2k} = A \cdot \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} u^{2k} + B \cdot \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} v^{2k} + C \cdot \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} w^{2k} =$$

$$= A(au^2 + b)^n + B(av^2 + b)^n + C(aw^2 + b)^n = Au^{3n} + Bv^{3n} + Cw^{3n} = x_{3n}, \forall n \in \mathbb{N}.$$

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