R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**Prove that:**

$$\psi\left(\frac{5}{8}\right)-\psi\left(\frac{1}{8}\right)=\pi\sqrt{2}+2\sqrt{2}\log\left(1+\sqrt{2}\right)$$

**where $\psi(*)$ –is the digamma function.**

*Proposed by Vasile Mircea Popa-Romania*

*Solution 1 by Probal Chakraborty-India, Solution 2 by Akerele Olofin-Nigeria, Solution 3 by Arslan Ahmed-Yemen, Solution 4 by Serlea Kabay-Liberia, Solution 5 by Mohammad Rostami-Kabul-Afghanistan*

*Solution 1 by Probal Chakraborty-India*

$$\because \psi(z+1)=-\gamma+\int_0^1\frac{1-t^z}{1-t}dt$$

$$\psi\left(1-\frac{3}{8}\right)=-\gamma+\int_0^1\frac{1-t^{-\frac{7}{8}}}{1-t}dt;\ \ \psi\left(\frac{5}{8}\right)=-\gamma+\int_0^1\frac{1-t^{-\frac{7}{8}}}{1-t}dt$$

$$\psi\left(\frac{5}{8}\right)-\psi\left(\frac{1}{8}\right)=\int_0^1\frac{1-t^{-\frac{3}{8}}-1+t^{-\frac{7}{8}}}{1-t}dt=\int_0^1\frac{t^{-\frac{7}{8}}-t^{-\frac{3}{8}}}{1-t}dt\overset{z^8=t}{=}8\int_0^1\frac{z^{-7}-z^{-3}}{1-z^8}z^7dz=$$

$$=8\int_0^1\frac{1-z^4}{1-z^8}dz=8\left(\int_0^1\frac{1}{1+z^4}dz+\int_0^1\frac{\frac{1}{z^2}}{z^2+\frac{1}{z^2}}dz\right)=\frac{8}{2}\int_0^1\frac{\frac{1}{z^2}-1+1+\frac{1}{z^2}}{z^2+\frac{1}{z^2}}dz=$$

$$= \frac{8}{2}\left(\int_0^1 \frac{d\left(z-\frac{1}{z}\right)}{\left(z-\frac{1}{z}\right)^2+2}\right) = -\frac{8}{2}\int_0^1 \frac{d\left(z+\frac{1}{z}\right)}{\left(z+\frac{1}{z}\right)^2-2} =$$

$$= \frac{8}{2\sqrt{2}}tan^{-1}\left|\frac{z-\frac{1}{z}}{\sqrt{2}}\right|_0^1 + \frac{8}{2\sqrt{2}}log\left|\frac{z+\frac{1}{z}-\sqrt{2}}{z+\frac{1}{z}+\sqrt{2}}\right|_0^1 = \pi\sqrt{2}+2\sqrt{2}log(1+\sqrt{2})$$

**Solution 2 by Akerele Olofin-Nigeria**

$$\psi\left(\frac{r}{m}\right) = -\gamma - log(2m) - \frac{\pi}{2}cot\left(\frac{r\pi}{m}\right) + 2\sum_{n=1}^{\left[\frac{m-1}{2}\right]} cos\left(\frac{2\pi nr}{m}\right)log\left(sin\left(\frac{\pi n}{m}\right)\right),$$

$r < m$, $[*] - GIF$. Now,

$$\psi\left(\frac{5}{8}\right) = -\gamma - log16 - \frac{\pi}{2}cot\left(\frac{5\pi}{8}\right) + 2\sum_{n=1}^{3} cos\left(\frac{10n\pi}{8}\right)log\left(sin\left(\frac{\pi n}{8}\right)\right) =$$

$$= -\gamma - 4log2 - \frac{\pi}{2}(1-\sqrt{2}) + \frac{2}{\sqrt{2}}log(1+\sqrt{2}) =$$

$$= -\gamma - 4log2 - \frac{\pi}{2} + \frac{\sqrt{2}}{2}\pi + \frac{2}{\sqrt{2}}log(1+\sqrt{2})$$

$$\psi\left(\frac{1}{8}\right) = -\gamma - log16 - \frac{\pi}{2}cot\left(\frac{\pi}{8}\right) + 2\sum_{n=1}^{3} cos\left(\frac{2n\pi}{8}\right)log\left(sin\left(\frac{\pi n}{8}\right)\right) =$$

$$= -\gamma - 4log2 - \frac{\pi}{2}(1+\sqrt{2}) + \frac{2}{\sqrt{2}}log(\sqrt{2}-1) =$$

$$= -\gamma - 4log2 - \frac{\pi}{2} - \frac{\sqrt{2}}{2}\pi + \frac{2}{\sqrt{2}}log(\sqrt{2}-1) \Rightarrow$$

$$\psi\left(\frac{5}{8}\right) - \psi\left(\frac{1}{8}\right) =$$

$$= -\gamma - 4log2 - \frac{\pi}{2} + \frac{\sqrt{2}}{2}\pi + \frac{2}{\sqrt{2}}log(1+\sqrt{2}) - \left(-\gamma - 4log2 - \frac{\pi}{2} - \frac{\sqrt{2}}{2}\pi + \frac{2}{\sqrt{2}}log(\sqrt{2}-1)\right)$$

$$= \sqrt{2}\pi + \frac{2}{\sqrt{2}}\left(log(1+\sqrt{2}) - log(\sqrt{2}-1)\right) = \pi\sqrt{2}+2\sqrt{2}log(1+\sqrt{2})$$

Therefore,

$$\psi\left(\frac{5}{8}\right)-\psi\left(\frac{1}{8}\right)=\pi\sqrt{2}+2\sqrt{2}log\left(1+\sqrt{2}\right)$$

## Solution 3 by Arslan Ahmed-Yemen

$$\Omega=\psi\left(\frac{5}{8}\right)-\psi\left(\frac{1}{8}\right)=-\int_0^1\frac{x^{-\frac{3}{8}}-x^{-\frac{7}{8}}}{1-x}dx\overset{u^8=x}{=}8\int_0^1\frac{du}{1+u^4}=$$

$$=\frac{4}{\sqrt{2}}\int_0^1\frac{u+\sqrt{2}}{u^2+\sqrt{2}u+1}du-\frac{4}{\sqrt{2}}\int_0^1\frac{u-2}{u^2-\sqrt{2}u+1}du=$$

$$=\left[\sqrt{2}log\left(u^2+\sqrt{2}u+1\right)-\sqrt{2}log\left(u^2-\sqrt{2}u+1\right)\right]_0^1-$$

$$-\left[2\sqrt{2}tan^{-1}\left(\sqrt{2}u+1\right)+2\sqrt{2}tan^{-1}\left(\sqrt{2}u-1\right)\right]_0^1=\sqrt{2}(log\left(\frac{\left(2+\sqrt{2}\right)^2}{2}+\pi\right)$$

**Therefore,**

$$\psi\left(\frac{5}{8}\right)-\psi\left(\frac{1}{8}\right)=\pi\sqrt{2}+2\sqrt{2}log\left(1+\sqrt{2}\right)$$

## Solution 4 by Serlea Kabay-Liberia

Recall $\psi(x)=-\gamma+\sum_{k=1}^{\infty}\frac{x}{n(n+x)}$. Let

$$\omega=\psi\left(\frac{5}{8}\right)-\psi\left(\frac{1}{8}\right)=-\gamma+\gamma+\sum_{n=1}^{\infty}\frac{\frac{5}{8}}{n\left(n+\frac{5}{8}\right)}-\sum_{n=1}^{\infty}\frac{\frac{1}{8}}{n\left(n+\frac{1}{8}\right)}$$

$$\omega=\frac{1}{8}\sum_{n=1}^{n}\left(\frac{5}{n\left(n+\frac{5}{8}\right)}-\frac{1}{n\left(n+\frac{1}{8}\right)}\right)=\frac{1}{8}\sum_{n=1}^{\infty}\left(\frac{64}{8n+1}-\frac{64}{8n+5}\right)=$$

$$=8\sum_{n=1}^{\infty}\left(\frac{1}{8n+1}-\frac{1}{8n+5}\right)=8\int_0^1\frac{dx}{1+x^4}=$$

$$=\frac{8}{2\sqrt{2}}\int_0^1\left(\frac{x+\sqrt{2}}{x^2+x\sqrt{2}+1}-\frac{x-\sqrt{2}}{x^2-x\sqrt{2}+1}\right)dx=$$

$$=\frac{2}{\sqrt{2}}log\left(2+\sqrt{2}\right)-\frac{1}{\sqrt{2}}log\left(2-\sqrt{2}\right)+\frac{2}{\sqrt{2}}\int_0^1\frac{\sqrt{2}}{x^2+x\sqrt{2}+1}dx+$$

$$+\frac{2}{\sqrt{2}}\int_0^1\frac{\sqrt{2}}{x^2-x\sqrt{2}+1}dx=\frac{2}{\sqrt{2}}log\left(3+2\sqrt{2}\right)+\frac{2\pi}{\sqrt{2}}$$

Therefore,

$$\psi\left(\frac{5}{8}\right)-\psi\left(\frac{1}{8}\right)=\pi\sqrt{2}+2\sqrt{2}log(1+\sqrt{2})$$

***Solution 5 by Mohammad Rostami-Kabul-Afghanistan***

$$\because \sum_{n=0}^{\infty}\frac{1}{n+a}=-\psi(a),\qquad \Phi(z,s,a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-at}}{1-ze^{-t}}dt$$

$$\psi\left(\frac{5}{8}\right)-\psi\left(\frac{1}{8}\right)=-\sum_{n=0}^{\infty}\frac{1}{n+\frac{5}{8}}+\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{8}}=-\Phi\left(1,1,\frac{5}{8}\right)+\Phi\left(1,1,\frac{1}{8}\right)=$$

$$=-\frac{1}{\Gamma(1)}\int_0^{\infty}\frac{e^{-\frac{5}{8}t}}{1-e^{-t}}dt+\frac{1}{\Gamma(1)}\int_0^{\infty}\frac{e^{-\frac{1}{8}t}}{1-e^{-t}}dt\overset{e^{-t}=u}{=}$$

$$=-\int_1^0\frac{u^{\frac{5}{8}}}{1-u}\left(-\frac{du}{u}\right)+\int_1^0\frac{u^{\frac{1}{8}}}{1-u}\left(-\frac{du}{u}\right)=\int_0^1\frac{-u^{-\frac{3}{8}}+u^{-\frac{7}{8}}}{1-u}du\overset{y^8=u}{=}$$

$$=\int_0^1\frac{-y^{-3}+y^{-7}}{1-y^8}(8y^7dy)=8\int_0^1\frac{1-y^4}{1-y^8}dy=8\int_0^1\frac{1}{1+y^4}dy=$$

$$=-4\int_0^1\frac{-\frac{2}{y^2}}{\frac{1}{y^2}+y^2}dy=-4\int_0^1\frac{\left(-\frac{1}{y^2}+1\right)+\left(-\frac{1}{y^2}-1\right)}{\frac{1}{y^2}+y^2}dy=$$

$$=-4\int_0^1\frac{d\left(\frac{1}{y}+y\right)}{\left(\frac{1}{y}+y\right)^2-2}-4\int_0^1\frac{d\left(\frac{1}{y}-y\right)}{\left(\frac{1}{y}-y\right)^2+2}=$$

$$=-4\left[\frac{1}{2\sqrt{2}}log\left(\frac{\frac{1}{y}+y-\sqrt{2}}{\frac{1}{y}+y+\sqrt{2}}\right)\right]_0^1-4\left[\frac{1}{\sqrt{2}}tan^{-1}\left(\frac{\frac{1}{y}-y}{\sqrt{2}}\right)\right]_0^1=$$

$$=\frac{2}{\sqrt{2}}log\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)+\frac{4}{\sqrt{2}}\cdot\frac{\pi}{2}=\frac{2}{\sqrt{2}}\pi+\frac{2}{\sqrt{2}}log\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)=\pi\sqrt{2}+2\sqrt{2}log(1+\sqrt{2})$$

Therefore,

$$\psi\left(\frac{5}{8}\right)-\psi\left(\frac{1}{8}\right)=\pi\sqrt{2}+2\sqrt{2}log(1+\sqrt{2})$$

**Note by editor:**

**Many thanks to Florică Anastase-Romania for typed solutions.**