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ABOUT AN INEQUALITY BY MARIAN URSĂRESCU INEQUALITY-XII

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1) In $\Delta ABC, AA', BB', CC'$ −internal bisectors, $\Delta A"B"C"$ −circumcevian triangle of incenter. Prove that:

$$\frac{[A'B'C']}{[A"B"C"]} \le \frac{r}{2R}$$

Proposed by Marian Ursărescu-Romania

Solution. **Lemma. 2) If AA', BB', CC' −internal bisectors in $\Delta ABC, \Delta A"B"C"$ −circumcevian triangle of incenter, then:**

$$\frac{[A'B'C']}{[A"B"C"]} = \frac{8r^2}{s^2 + r^2 + 2Rr}$$

Proof. We have

$$[A"B"C"] = \frac{2abc}{(a+b)(b+c)(c+a)} \cdot F = \frac{2 \cdot 4Rrs}{2s(s^2 + r^2 + 2Rr)} \cdot rs = \frac{4Rr^2 s}{s^2 + r^2 + 2Rr}; (1)$$

In $\Delta A"B"C"$ we have $\angle A" = \frac{\pi}{2} - \frac{A}{2}$ and $B''C'' = 2Rcos\frac{A}{2}$, thus

$$[A''B''C''] = \frac{2Rcos\frac{A}{2} \cdot 2Rcos\frac{B}{2} \cdot cos\frac{C}{2}}{4R} = 2R^2 \prod_{cyc} cos\frac{A}{2} = 2R^2 \cdot \frac{s}{4R} = \frac{sR}{2}; (2)$$

From (1),(2) it follows that: $\frac{[A'B'C']}{[A"B"C"]} = \frac{8r^2}{s^2+r^2+2Rr}$.Let's get back to the main problem.

Using Lemma, inequality can be written as: $\frac{8r^2}{s^2+r^2+2Rr} \le \frac{r}{2R} \Leftrightarrow s^2 \ge 14Rr - r^2$, which follows from Gerretsen inequality $s^2 \ge 16Rr - 5r^2$ and $R \ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

3) In $\Delta ABC, AA', BB', CC'$ −internal bisectors, $\Delta A"B"C"$ −circumcevian triangle of incenter. Prove that:

$$\frac{[A'B'C']}{[A"B"C"]} \ge \frac{r^2}{R^2}$$

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Solution. Using Lemma, inequality can be written as: $\frac{8r^2}{s^2+r^2+2Rr} \geq \frac{r^2}{R^2} \Leftrightarrow$

$s^2 \leq 8R^2 - 2Rr - r^2$, which follows from $s^2 \leq 4R^2 + 4Rr + 3r^2 (Gerretsen)$ and $R \geq 2r (Euler)$. Equality holds if and only if triangle is equilateral.

4) In $\Delta ABC, AA', BB', CC'$ −internal bisectors, $\Delta A"B"C"$ −circumcevian triangle of incenter. Prove that:

$$\frac{r}{2R} \leq \frac{[A'B'C']}{[A"B"C"]} \geq \frac{r^2}{R^2}$$

Proposed by Marin Chirciu-Romania

Solution. See inequalities 1), 3). Equality holds if and only if triangle is equilateral.

REFERENCE:

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