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ABOUT AN INEQUALITY BY KOSTAS GERONIKOLAS-III

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1) In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{r_a}{m_b m_c}} \le \frac{1}{r}\sqrt{\frac{3R}{2}}$$

Proposed by Kostas Geronikolas-Greece

Solution. **Lemma. 2) In ΔABC the following relationship holds:**

$$\left(\sum_{cyc}\sqrt{\frac{r_a}{m_b m_c}}\right)^2 \le \frac{1}{r}\left(\frac{4R+r}{s}\right)^2$$

Proof. Using BCS inequality, we get:

$$\left(\sum_{cyc}\sqrt{\frac{r_a}{m_b m_c}}\right)^2 \le \sum_{cyc} r_a \cdot \sum_{cyc}\frac{1}{m_b m_c} = (4R+r)\sum_{cyc}\frac{1}{m_b m_c} = \frac{4R+r}{m_a m_b m_c}\sum_{cyc} m_a \le$$

$$\le \frac{4R+r}{r_a r_b r_c}(4R+r) = \frac{(4R+r)^2}{rs^2} = \frac{1}{r}\left(\frac{4R+r}{s}\right)^2$$

Let's get back to the main problem. Using Lemma, it is enough to prove that:

$\frac{1}{r}\left(\frac{4R+r}{s}\right)^2 \le \frac{3R}{2r^2} \Leftrightarrow \frac{(4R+r)^2}{s^2} \le \frac{3R}{2r} \Leftrightarrow 2r(4R+r)^2 \le 3Rs^2$, which follows from

$s^2 \ge 16Rr - 5r^2 \ge \frac{r(4R+r)^2}{R+r}(Gerretsen)$ and $R \ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

3) In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{r_a}{m_b m_c}} \le \frac{4R+r}{s}\sqrt{\frac{1}{r}}$$

Proposed by Marin Chirciu-Romania

Solution. Using BCS inequality, we obtain:

$$\left(\sum_{cyc}\sqrt{\frac{r_a}{m_bm_c}}\right)^2 \le \sum_{cyc} r_a \cdot \sum_{cyc}\frac{1}{m_bm_c} = (4R+r)\sum_{cyc}\frac{1}{m_bm_c} = \frac{4R+r}{m_am_bm_c}\sum_{cyc} m_a \le$$

$$\le \frac{4R+r}{r_ar_br_c}(4R+r) = \frac{(4R+r)^2}{rs^2} = \frac{1}{r}\left(\frac{4R+r}{s}\right)^2$$

Equality holds if and only if triangle is equilateral.

4) In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{r_a}{m_bm_c}} \le \frac{4R+r}{s}\sqrt{\frac{1}{r}} \le \frac{1}{r}\sqrt{\frac{3R}{2}}$$

Proposed by Marin Chirciu-Romania

Solution. See inequality (3) and $\frac{4R+r}{s}\sqrt{\frac{1}{r}} \le \frac{1}{r}\sqrt{\frac{3R}{2}} \Leftrightarrow \frac{1}{r}\left(\frac{4R+r}{s}\right)^2 \le \frac{3R}{2r^2} \Leftrightarrow \frac{(4R+r)^2}{s^2} \le \frac{3R}{2r} \Leftrightarrow$

$2r(4R+r)^2 \le 3Rs^2$, which follows from $s^2 \ge 16Rr - 5r^2 \ge \frac{r(4R+r)^2}{R+r}(Gerretsen)$ and $R \ge 2r(Euler)$. Equality holds if and only if triangle is equilateral.

5) In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{r_a}{m_bm_c}} \ge 3\sqrt[6]{\frac{4r}{R^2s^2}}$$

Proposed by Marin Chirciu-Romania

Solution. Using AM-GM inequality, we get:

$$\sum_{cyc}\sqrt{\frac{r_a}{m_bm_c}} \ge 3\sqrt[3]{\prod_{cyc}\sqrt{\frac{r_a}{m_bm_c}}} = 3\sqrt[6]{\prod_{cyc}\frac{r_a}{m_bm_c}} = 3\sqrt[6]{\frac{\prod r_a}{\prod m_a^2}} \ge$$

$\ge 3\sqrt[6]{\frac{rs^2}{\left(\frac{Rs^2}{2}\right)^2}} = 3\sqrt[6]{\frac{rs^2}{\frac{R^2s^4}{4}}} = 3\sqrt[6]{\frac{4r}{R^2s^2}}$, which follows from $m_am_bm_c \le \frac{Rs^2}{2}$.

Lemma. In ΔABC the following relationship holds:

$$m_am_bm_c \le \frac{Rs^2}{2}$$

Proof. Using identity in triangle:

$$\prod_{cyc} m_a^2 = \frac{s^6 + s^4(33r^2 - 12Rr) - s^2r^2(60R^2 + 120Rr + 33r^2) - r^3(4R + r)^3}{16}$$

Inequality becomes:

$$\frac{s^6 + s^4(33r^2 - 12Rr) - s^2r^2(60R^2 + 120Rr + 33r^2) - r^3(4R + r)^3}{16} \le \left(\frac{Rs^2}{2}\right)^2 \Leftrightarrow$$

$$s^6 + s^4(33r^2 - 12Rr - 4R^2) - s^2r^2(60R^2 + 120Rr + 33r^2) - r^3(4R + r)^3 \le 0; (1)$$

Using $s^2 \le 4R^2 + 4Rr + 3r^2 (Gerretsen)$, we get:

$s^6 + s^4(33r^2 - 12Rr - 4R^2) \le s^4(36r^2 - 8Rr)$; (2), which follows from

$$s^6 + s^4(33r^2 - 12Rr - 4R^2) = s^4(s^2 + 33r^2 - 12Rr - 4R^2) \le$$
$$\le s^4(4R^2 + 4Rr + 3r^2 + 33r^2 - 12Rr - 4r^2) \le s^4(36r^2 - 8Rr).$$

From(1),(2) it is enough to prove that:

$$s^4(36r^2 - 8Rr) - s^2r^2(60R^2 + 120Rr + 33r^2) - r^3(4R + r)^3 \le 0 \Leftrightarrow$$
$$s^4(36r - 8R) - s^2r(60R^2 + 120Rr + 33r^2) - r^2(4R + r)^3 \le 0 \Leftrightarrow$$

$s^4(8R - 16r) + s^2r(60R^2 + 120Rr + 33r^2) + r^2(4R + r)^3 \ge 20rs^4$, which follows from $16Rr - 5r^2 \le s^2 \le 4R^2 + 4Rr + 3r^2 (Gerretsen)$. Remains to prove that:

$$s^2(16Rr - 5r^2)(8R - 16r) + s^2r(60R^2 + 120Rr + 33r^2) + r^2(4R + r)^3$$
$$\ge 20rs^2(4R^2 + 4Rr + 3r^2) \Leftrightarrow$$
$$s^2(16R - 5r)(8R - 16r) + s^2(60R^2 + 120Rr + 33r^2) + r(4R + r)^2$$
$$\ge 20s^2(4R^2 + 4Rr + 3r^2) \Leftrightarrow$$

$s^2(108R^2 - 256Rr + 53r^2) + r(4R + r)^2 \ge 0$. Distinguish the cases:

Case 1) If $(108R^2 - 256Rr + 53r^2) \ge 0$ inequality is obviously true.

Case 2) If $(108R^2 - 256Rr + 53r^2) < 0$ inequality can be written as:

$r(4R + r)^3 \ge -s^2(108R^2 - 256Rr + 53r^2)$, which follows from

$$s^2 \le \frac{R(4R + r)^2}{2(2R - r)} \le 4R^2 + 4Rr + 3r^2 (Blundon - Gerretsen)$$

Remains to prove that: $r(4R + r)^3 \ge \frac{r(4R+r)^2}{2(2R-r)}(-108R^2 + 256R - 53r^2) \Leftrightarrow$

$$2r(2R - r)(4R + r) \ge R(-108R^2 + 256Rr - 53r^2) \Leftrightarrow$$

$108R^3 - 240R^2r + 49Rr^2 - 2r^3 \ge 0 \Leftrightarrow (R - 2r)(108R^2 - 24Rr + r^2) \ge 0$, which is true from $R \ge 2r (Euler)$.

Equality holds if and only of triangle is equilateral.

6) In ΔABC the following relationship holds:

$$3\sqrt[6]{\frac{4r}{R^2s^2}} \le \sum_{cyc} \sqrt{\frac{r_a}{m_b m_c}} \le \frac{4R + r}{s}\sqrt{\frac{1}{r}}$$

Proposed by Marin Chirciu-Romania

Solution. For RHS, applying BCS inequality, we get:

$$\left(\sum_{cyc}\sqrt{\frac{r_a}{m_bm_c}}\right)^2\le\sum_{cyc}r_a\cdot\sum_{cyc}\frac{1}{m_bm_c}=(4R+r)\sum_{cyc}\frac{1}{m_bm_c}=\frac{4R+r}{m_am_bm_c}\sum_{cyc}m_a\le$$

$$\le\frac{4R+r}{r_ar_br_c}(4R+r)=\frac{(4R+r)^2}{rs^2}=\frac{1}{r}\left(\frac{4R+r}{s}\right)^2$$

Equality holds if and only if triangle is equilateral. For LHS, applying AM-GM inequality, we get:

$$\sum_{cyc}\sqrt{\frac{r_a}{m_bm_c}}\ge 3\sqrt[3]{\prod_{cyc}\sqrt{\frac{r_a}{m_bm_c}}}=3\sqrt[6]{\prod_{cyc}\frac{r_a}{m_bm_c}}=3\sqrt[6]{\frac{\prod r_a}{\prod m_a^2}}\ge$$

$$\ge 3\sqrt[6]{\frac{rs^2}{\left(\frac{Rs^2}{2}\right)^2}}=3\sqrt[6]{\frac{rs^2}{\frac{R^2s^4}{4}}}=3\sqrt[6]{\frac{4r}{R^2s^2}},$$ which follows from $m_am_bm_c\le\frac{Rs^2}{2}$.

Lemma. In ΔABC the following relationship holds:

$$m_am_bm_c\le\frac{Rs^2}{2}$$

Proof. Using identity in triangle:

$$\prod_{cyc}m_a^2=\frac{s^6+s^4(33r^2-12Rr)-s^2r^2(60R^2+120Rr+33r^2)-r^3(4R+r)^3}{16}$$

Inequality becomes:

$$\frac{s^6+s^4(33r^2-12Rr)-s^2r^2(60R^2+120Rr+33r^2)-r^3(4R+r)^3}{16}\le\left(\frac{Rs^2}{2}\right)^2\Leftrightarrow$$

$$s^6+s^4(33r^2-12Rr-4R^2)-s^2r^2(60R^2+120Rr+33r^2)-r^3(4R+r)^3\le 0;(1)$$

Using $s^2\le 4R^2+4Rr+3r^2(Gerretsen)$, we get:

$s^6+s^4(33r^2-12Rr-4R^2)\le s^4(36r^2-8Rr)$; (2), which follows from

$$s^6+s^4(33r^2-12Rr-4R^2)=s^4(s^2+33r^2-12Rr-4R^2)\le$$

$$\le s^4(4R^2+4Rr+3r^2+33r^2-12Rr-4r^2)\le s^4(36r^2-8Rr).$$

From(1),(2) it is enough to prove that:

$$s^4(36r^2-8Rr)-s^2r^2(60R^2+120Rr+33r^2)-r^3(4R+r)^3\le 0\Leftrightarrow$$

$$s^4(36r-8R)-s^2r(60R^2+120Rr+33r^2)-r^2(4R+r)^3\le 0\Leftrightarrow$$

$s^4(8R-16r)+s^2r(60R^2+120Rr+33r^2)+r^2(4R+r)^3\ge 20rs^4$, which follows from $16Rr-5r^2\le s^2\le 4R^2+4Rr+3r^2(Gerretsen)$. Remains to prove that:

$$s^2(16Rr-5r^2)(8R-16r)+s^2r(60R^2+120Rr+33r^2)+r^2(4R+r)^3\ge 20rs^2(4R^2+4Rr+3r^2)\Leftrightarrow$$

$$s^2(16R-5r)(8R-16r)+s^2(60R^2+120Rr+33r^2)+r(4R+r)^2\ge 20s^2(4R^2+4Rr+3r^2)\Leftrightarrow$$

$s^2(108R^2-256Rr+53r^2)+r(4R+r)^2\ge 0$. Distinguish the cases:

Case 1) If $(108R^2-256Rr+53r^2)\ge 0$ inequality is obviously true.

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$r(4R + r)^3 \geq -s^2(108R^2 - 256Rr + 53r^2)$, which follows from

$$s^2 \leq \frac{R(4R + r)^2}{2(2R - r)} \leq 4R^2 + 4Rr + 3r^2(Blundon - Gerretsen)$$

Remains to prove that: $r(4R + r)^3 \geq \frac{r(4R+r)^2}{2(2R-r)}(-108R^2 + 256R - 53r^2) \Leftrightarrow$

$2r(2R - r)(4R + r) \geq R(-108R^2 + 256Rr - 53r^2) \Leftrightarrow$

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