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ABOUT AN INEQUALITY BY ERTAN YILDIRIM-XI

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1) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{r_a(r_b^3+r_c^3)}{a}\ge s^3$$

Proposed by Ertan Yildirim-Turkey

Solution. **Lemma. 2) In ΔABC the following relationship holds:**

$$\sum_{cyc}\frac{r_a(r_b^3+r_c^3)}{a}=s[2(4R+r)^2-5s^2]$$

Proof. Using identity: $r_a=\frac{F}{s-a}$, we get:

$$\sum_{cyc}\frac{r_a(r_b^3+r_c^3)}{a}=\sum_{cyc}\frac{\frac{F}{s-a}\left(\left(\frac{F}{s-b}\right)^3+\left(\frac{F}{s-c}\right)^3\right)}{a}=F^4\sum_{cyc}\frac{(s-b)^3+(s-c)^3}{a(s-a)(s-b)^3(s-c)^3}=$$

$$=\frac{F^4}{(s-a)(c-b)(s-c)}\sum_{cyc}\frac{(s-b)^3+(s-c)^3}{a(s-b)^3(s-c)^3}=\frac{r^4s^4}{r^2s}\sum_{cyc}\frac{(s-b)^3+(s-c)^3}{a(s-b)^2(s-c)^2}=$$

$$=\frac{r^2s^3}{abc(s-a)^2(s-b)^2(s-c)^2}\sum_{cyc}bc(s-a)^2[(s-b)^3+(s-c)^3]=$$

$$=\frac{r^2s^3}{4Rrs(r^2s)^2}\sum_{cyc}bc(s-a)^2[(s-b)^3+(s-c)^3]=$$

$=\frac{1}{4Rr^3s}\cdot 4Rr^3s\,[2(4R+r)^2-5s^2]=s[2(4R+r)^2-5s^2]$, which follows from

$$\sum_{cyc}bc(s-a)^2[(s-b)^3+(s-c)^3]=4Rr^3s\,[2(4R+r)^2-5s^2]$$

Let's get back to the main problem. Using Lemma, inequality can be written as:
$s[2(4R+r)^2-5s^2]\ge s^3\Leftrightarrow 2(4R+r)^2-5s^2\ge s^2\Leftrightarrow 2(4R+r)^2\ge 6s^2\Leftrightarrow$
$(4R+r)^2\ge 3s^2(Doucet)$, which follows from Gerretsen inequality
$s^2\le 4R^2+4Rr+3r^2$ and Euler inequality $R\ge 2r$.
Equality holds if and only if triangle is equilateral.

3) In ΔABC the following relationship holds:

$\sum_{cyc}\frac{r_a(r_b^3+r_c^3)}{a}\ge\frac{27}{4}R^2s.$ ***Proposed by Marin Chirciu-Romania***

Solution. Using Lemma and Gerretsen inequality: $s^2 \le 4R^2 + 4Rr + 3r^2$, it follows that:

$$s[2(4R+r)^2 - 5s^2] \ge s[2(4R+r)^2 - 5(4R^2 + 4Rr + 3r^2)] =$$
$$= s[32R^2 + 16Rr + 2r^2 - 20R^2 - 20Rr - 15r^2] =$$
$$= s(12R^2 - 4Rr - 13r^2) \overset{Euler}{\ge} s \cdot \frac{27R^2}{4} = \frac{27}{4}R^2 s$$

Equality holds if and only if triangle is equilateral.

4) In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{r_a(r_b^3 + r_c^3)}{a} \ge \frac{27}{4}R^2 s \ge s^3$$

Proposed by Marin Chirciu-Romania

Solution. See inequality 3) and $\frac{27}{4}R^2 s \ge s^3 \Leftrightarrow \frac{27}{4}R^2 \ge s^2 (Mitrinovic)$.

Equality holds if and only if triangle is equilateral.

5) In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{r_a(r_b^3 + r_c^3)}{a} \le s(32R^2 - 101r^2)$$

Proposed by Marin Chirciu-Romania

Solution. Using Lemma and Gerretsen inequality: $s^2 \ge 16Rr - 5r^2$, it follows that:

$$s[2(4R+r)^2 - 5s^2] \le s[2(4R+r)^2 - 5(16Rr - 5r^2)] \Leftrightarrow$$
$$s[2(4R+r)^2 - 5s^2] \le s[2(4R+r)^2 - 5(16Rr - 5r^2)] =$$
$$= s[2(16R^2 + 8Rr + r^2) - 5(16Rr - 5r^2)] =$$
$$= s(32R^2 + 16Rr + 2r^2 - 80rr + 25r^2) = s(32R^2 - 64Rr + 27r^2) \overset{Euler}{\le}$$

$s(32R^2 - 101r^2) \Leftrightarrow (4R+r)^2 \ge 3s^2 (Doucet)$, which follows from Gerretsen inequality $s^2 \le 4R^2 + 4Rr + 3r^2$ and Euler inequality $R \ge 2r$.

Equality holds if and only if triangle is equilateral.

6) In ΔABC the following relationship holds:

$$\frac{27}{4}R^2 s \le \sum_{cyc} \frac{r_a(r_b^3 + r_c^3)}{a} \le s(32R^2 - 101r^2)$$

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Solution. **Lemma. 7) In ΔABC the following relationship holds:**

$$\sum_{cyc}\frac{r_a(r_b^3+r_c^3)}{a}=s[2(4R+r)^2-5s^2]$$

Proof. Using identity: $r_a=\frac{F}{s-a}$, we get:

$$\sum_{cyc}\frac{r_a(r_b^3+r_c^3)}{a}=\sum_{cyc}\frac{\frac{F}{s-a}\left(\left(\frac{F}{s-b}\right)^3+\left(\frac{F}{s-c}\right)^3\right)}{a}=F^4\sum_{cyc}\frac{(s-b)^3+(s-c)^3}{a(s-a)(s-b)^3(s-c)^3}=$$

$$=\frac{F^4}{(s-a)(c-b)(s-c)}\sum_{cyc}\frac{(s-b)^3+(s-c)^3}{a(s-b)^3(s-c)^3}=\frac{r^4s^4}{r^2s}\sum_{cyc}\frac{(s-b)^3+(s-c)^3}{a(s-b)^2(s-c)^2}=$$

$$=\frac{r^2s^3}{abc(s-a)^2(s-b)^2(s-c)^2}\sum_{cyc}bc(s-a)^2[(s-b)^3+(s-c)^3]=$$

$$=\frac{r^2s^3}{4Rrs(r^2s)^2}\sum_{cyc}bc(s-a)^2[(s-b)^3+(s-c)^3]=$$

$=\frac{1}{4Rr^3s}\cdot 4Rr^3s\,[2(4R+r)^2-5s^2]=s[2(4R+r)^2-5s^2]$, which follows from

$$\sum_{cyc}bc(s-a)^2[(s-b)^3+(s-c)^3]=4Rr^3s\,[2(4R+r)^2-5s^2]$$

Let's get back to the main problem. For RHS using Lemma and Gerretsen inequality: $s^2\ge 16Rr-5r^2$, it follows that:

$$s[2(4R+r)^2-5s^2]\le s[2(4R+r)^2-5(16Rr-5r^2)]\Leftrightarrow$$

$$s[2(4R+r)^2-5s^2]\le s[2(4R+r)^2-5(16Rr-5r^2)]=$$

$$=s[2(16R^2+8Rr+r^2)-5(16Rr-5r^2)]=$$

$$=s(32R^2+16Rr+2r^2-80rr+25r^2)=s(32R^2-64Rr+27r^2)\overset{Euler}{\le}$$

$s(32R^2-101r^2)\Leftrightarrow(4R+r)^2\ge 3s^2(Doucet)$, which follows from Gerretsen inequality $s^2\le 4R^2+4Rr+3r^2$ and Euler inequality $R\ge 2r$.

Equality holds if and only if triangle is equilateral.

For LHS Using Lemma and Gerretsen inequality: $s^2\le 4R^2+4Rr+3r^2$, it follows that:

$$s[2(4R+r)^2-5s^2]\ge s[2(4R+r)^2-5(4R^2+4Rr+3r^2)]=$$

$$=s[32R^2+16Rr+2r^2-20R^2-20Rr-15r^2]=$$

$$=s(12R^2-4Rr-13r^2)\overset{Euler}{\ge}s\cdot\frac{27R^2}{4}=\frac{27}{4}R^2s$$

Equality holds if and only if triangle is equilateral.

8) In ΔABC the following relationship holds:

$$\frac{s}{3}\left(\frac{2r}{R}\right)^3(4R+r)^2\le\sum_{cyc}\frac{h_a(h_b^3+h_c^3)}{a}\le\frac{s}{3}\left(\frac{2r}{R}\right)(4R+r)^2$$

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Solution. **Lemma. 9) In ΔABC the following relationship holds:**

$$\sum_{cyc}\frac{h_a(h_b^3+h_c^3)}{a}=\frac{rs}{2R^3}[s^2(s^2-4Rr)-r^2(4R+r)^2]$$

Proof. Using identity $h_a=\frac{2F}{a}$ we get:

$$\sum_{cyc}\frac{h_a(h_b^3+h_c^3)}{a}=\sum_{cyc}\frac{\frac{2F}{a}\left(\left(\frac{2F}{b}\right)^3+\left(\frac{2F}{c}\right)^3\right)}{a}=16F^4\sum_{cyc}\frac{\frac{1}{a}\left(\frac{1}{b^3}+\frac{1}{c^3}\right)}{a}=16F^4\sum_{cyc}\frac{b^3+c^3}{a^2b^3c^3}$$

$$=\frac{16F^4}{a^2b^2c^2}\sum_{cyc}\frac{b^3+c^3}{bc}=\frac{16r^4s^4}{16R^2r^2s^2}\sum_{cyc}\frac{a(b^3+c^3)}{abc}=\frac{r^2s^2}{R^2}\sum_{cyc}\frac{a(b^3+c^3)}{abc}=$$

$$=\frac{r^2s^2}{R^2abc}\sum_{cyc}a(b^3+c^3)=\frac{r^2s^2}{R^2\cdot 4Rrs}\cdot 2[s^4-4Rrs^2-(4R+r)^2]=$$

$=\frac{rs}{2R^3}[s^2(s^2-4Rr)-r^2(4R+r)^2]$, which follows from

$$\sum_{cyc}a(b^3+c^3)=2[s^4-4Rrs^2-(4R+r)^2]$$

For RHS using Lemma and Blundon-Gerretsen: $s^2\geq\frac{R(4R+r)^2}{2(2R-r)}\leq 4R^2+4Rr+3r^2$, we get:

$$\sum_{cyc}\frac{h_a(h_b^3+h_c^3)}{a}=\frac{rs}{2R^3}[s^2(s^2-4Rr)-r^2(4R+r)^2]\leq$$

$$\leq\frac{rs}{2R^3}\left[\frac{R(4R+r)^2}{2(2R-r)}(4R^2+4Rr+3r^2-4Rr)-r^2(4R+r)^2\right]=$$

$$=\frac{rs}{2R^2}(4R+r)^2\left[\frac{R}{2(2R-r)}(4R^2+3r^2)-r^2\right]=\frac{rs}{2R^3}(4R+r)^2\frac{4R^3-Rr^2+2r^2}{29(2R-r)}\leq$$

$$\overset{Euler}{\leq}\frac{rs}{2R^3}(4R+r)^2\frac{4R^2}{3}=\frac{s}{3}\left(\frac{2r}{R}\right)(4R+r)^2$$

Equality holds if and only if triangle is equilateral.

For LHS, using Lemma and Gerretsen inequality $s^2\geq 16Rr-5r^2\geq\frac{r(4R+r)^2}{R+r}$ we get:

$$\sum_{cyc}\frac{h_a(h_b^3+h_c^3)}{a}=\frac{rs}{2R^3}[s^2(s^2-4Rr)-r^2(4R+r)^2]\geq$$

$$\geq\frac{rs}{2R^3}\left[\frac{r(4R+r)^2}{R+r}(16Rr-5r^2-4Rr)-r^2(4R+r)^2\right]=$$

$$=\frac{rs}{2R^3}(4R+r)^2\left[\frac{r}{R+r}(12Rr-5r^2)-r^2\right]=\frac{rs}{2R^3}r^2(4R+r)^2\frac{11R-6r}{R+r}\overset{Euler}{\geq}$$

$$\geq\frac{rs}{2R^3}r^2(4R+r)^2\frac{16}{3}=\frac{s}{3}\left(\frac{2r}{R}\right)^3(4R+r)^2$$

Equality holds if and only if triangle is equilateral.

10) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{h_a\left(h_b^3+h_c^3\right)}{a}\le\sum_{cyc}\frac{r_a\left(r_b^3+r_c^3\right)}{a}$$

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Solution.

$$\sum_{cyc}\frac{h_a(h_b^3+h_c^3)}{a}=\frac{rs}{2R^3}[s^2(s^2-4Rr)-r^2(4R+r)^2]$$

$$\sum_{cyc}\frac{r_a(r_b^3+r_c^3)}{a}=s[2(4R+r)^2-5s^2]$$

Inequality can be written as:

$$\frac{rs}{2R^3}[s^2(s^2-4Rr)-r^2(4R+r)^2]\le s[2(4R+r)^2-5s^2]\Leftrightarrow$$

$s^2(rs^2+10R^3-4Rr^2)\le(4R+r)^2(4R^3+r^3)$, which follows from Blundon-Gerretsen

$s^2\le\frac{R(4R+r)^2}{2(2R-r)}\le 4R^2+4Rr+3r^2$. Remains to prove that:

$$\frac{R(4R+r)^2}{2(2R-r)}[r(4R^2+4Rr+3r^2)+10R^3-4Rr^2]\le(4R+r)^2(4R^3+r^3)\Leftrightarrow$$

$6R^4-12R^3r+Rr^3-2R^4\ge 0\Leftrightarrow(R-2r)(6R^3+r^3)\ge 0$, which is true from $R\ge 2r(Euler)$. Equality holds if and only if triangle is equilateral.

11) In ΔABC the following relationship holds:

$$\frac{s}{3}\left(\frac{2r}{R}\right)^3(4R+r)^2\le\sum_{cyc}\frac{h_a\left(h_b^3+h_c^3\right)}{a}\le\sum_{cyc}\frac{r_a\left(r_b^3+r_c^3\right)}{a}\le s(32R^2-101r^2)$$

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Solution. See inequalities 10), 8), and 6). Equality holds if and only if triangle is equilateral.

REFERENCE:

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