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ABOUT AN INEQUALITY BY ELDENIZ HESENOV-IV

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In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{AH\cdot AI}{\sqrt{r_b+r_c}}\right)^2\le 4R(R^2-3r^2)$$

Proposed by Eldeniz Hesenov-Georgia

Solution. **Lemma. 2) In ΔABC the following relationship holds:**

$$\sum_{cyc}\left(\frac{AH\cdot AI}{\sqrt{r_b+r_c}}\right)^2=\frac{R[s^4-2s^2r(4R+5r)+r^2(\$r=R+r)^2]}{s^2}$$

Proof. Using $AH=4R^2cos^2A=4R^2-a^2, AI=\frac{r^2}{sin^2\frac{A}{2}}=bc-4Rr$,

$r_b+r_c=\frac{F}{s-b}+\frac{F}{s-c}=4Rcos\frac{A}{2}$, it follows that:

$$\sum_{cyc}\left(\frac{AH\cdot AI}{\sqrt{r_b+r_c}}\right)^2=\sum_{cyc}\frac{AH^2\cdot AI^2}{r_b+r_c}=\sum_{cyc}\frac{(4R^2-a^2)(bc-4Rr)}{4Rcos^2\frac{A}{2}}=$$

$$=\frac{1}{4R}\sum_{cyc}\frac{(4R^2-a^2)(bc-4Rr)}{cos^2\frac{A}{2}}=\frac{1}{4R}\sum_{cyc}\frac{(4R^2-a^2)(bc-4Rr)}{\frac{s(s-a)}{bc}}=$$

$$=\frac{1}{4Rs}\cdot\frac{\sum bc(s-b)(s-c)(4R^2-a^2)(bc-4Rr)}{\prod(s-a)}=$$

$$=\frac{1}{4Rs}\cdot\frac{4R^2r^2[s^4-2s^2r(4R+5r)+r^2(4R+r)^2]}{r^2s}=$$

$=\frac{R[s^4-2s^2r(4R+5r)+r^2(4R+r)^2]}{s^2}$, which follows from

$$\sum_{cyc}b^2c^2(s-b)(s-c)=r^2[s^2(s^2+2r^2-4Rr)+r(4R+r)^3]$$

$$\sum_{cyc}bc(s-b)(s-c)=r^2[s^2+(4R+r)^2]$$

$$\sum_{cyc}a(s-b)(s-c)=2sr(2R-r)$$

$$\sum_{cyc}(s-b)(s-c)=r(4R+r)$$

Using Lemma, inequality can be written as:

$$\frac{R[s^4-2s^2r(4R+5r)+r^2(4R+r)^2]}{s^2}\le 4R(R^2-3r^2)\Leftrightarrow$$

$$R[s^4-2s^2r(4R+5r)+r^2(4R+r)^2]\le 4s^2(R^2-3r^2)\Leftrightarrow$$

$$s^4-2s^2r(4R+5r)+r^2(4R+r)^2\le 4s^2(R^2-3r^2)\Leftrightarrow$$

$s^2(4R^2+8Rr-2r^2-s^2)\ge r^2(4R+r)^2$, which follows from Blundon Gerretsen:

$$\frac{r(4R+r)^2}{R+r}\le 16Rr-5r^2\le 4R^2+4Rr+3r^2$$

Remains to prove that:

$$\frac{r(4R+r)^2}{R+r}(4R^2+8Rr-2r^2-4R^2-4Rr-3r^2)\ge r^2(4R+r)^2\Leftrightarrow$$

$$r(4R-5r)\ge r(R+r)\Leftrightarrow R\ge 2r(Euler).$$

Equality holds if and only if triangle is equilateral.

3) In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{AH\cdot AI}{\sqrt{r_b+r_c}}\right)^2\le R(4R^2-3Rr-6r^2)$$

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Solution. **Lemma. 4) In ΔABC the following relationship holds:**

$$\sum_{cyc}\left(\frac{AH\cdot AI}{\sqrt{r_b+r_c}}\right)^2=\frac{R[s^4-2s^2r(4R+5r)+r^2(\$r=R+r)^2]}{s^2}$$

Proof. Using $AH=4R^2cos^2A=4R^2-a^2, AI=\frac{r^2}{sin^2\frac{A}{2}}=bc-4Rr$,

$r_b+r_c=\frac{F}{s-b}+\frac{F}{s-c}=4Rcos\frac{A}{2}$, it follows that:

$$\sum_{cyc}\left(\frac{AH\cdot AI}{\sqrt{r_b+r_c}}\right)^2=\sum_{cyc}\frac{AH^2\cdot AI^2}{r_b+r_c}=\sum_{cyc}\frac{(4R^2-a^2)(bc-4Rr)}{4Rcos^2\frac{A}{2}}=$$

$$=\frac{1}{4R}\sum_{cyc}\frac{(4R^2-a^2)(bc-4Rr)}{cos^2\frac{A}{2}}=\frac{1}{4R}\sum_{cyc}\frac{(4R^2-a^2)(bc-4Rr)}{\frac{s(s-a)}{bc}}=$$

$$=\frac{1}{4Rs}\cdot\frac{\sum bc(s-b)(s-c)(4R^2-a^2)(bc-4Rr)}{\prod(s-a)}=$$

$$=\frac{1}{4Rs}\cdot\frac{4R^2r^2[s^4-2s^2r(4R+5r)+r^2(4R+r)^2]}{r^2s}=$$

$=\frac{R[s^4-2s^2r(4R+5r)+r^2(4R+r)^2]}{s^2}$, which follows from

$$\sum_{cyc}b^2c^2(s-b)(s-c)=r^2[s^2(s^2+2r^2-4Rr)+r(4R+r)^3]$$

$$\sum_{cyc} bc(s-b)(s-c) = r^2[s^2+(4R+r)^2]$$

$$\sum_{cyc} a(s-b)(s-c) = 2sr(2R-r)$$

$$\sum_{cyc} (s-b)(s-c) = r(4R+r)$$

Using Lemma, inequality can be written as:

$$\frac{R[s^4-2s^2r(4R+5r)+r^2(4R+r)^2]}{s^2} = R\left[s^2-2r(4R+5r)+\frac{r^2(4R+r)^2}{s^2}\right] \le$$

$$\le R\left[4R^2+4Rr+3r^2-2r(4R+5r)+\frac{r^2(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right] =$$

$= R[4R^2-4Rr-7r^2+r(R+r)] = R(4R^2-3Rr-6r^2)$, which follows from Gerretsen inequality: $\frac{r(4R+r)^2}{R+r} \le 16Rr-5r^2 \le 4R^2+4Rr+3r^2$.

Equality holds if and only if triangle is equilateral.

5) In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{AH\cdot AI}{\sqrt{r_b+r_c}}\right)^2 \le R(4R^2-3Rr-6r^2) \le 4R(R^2-3r^2)$$

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Solution. See inequality 3) and Euler inequality $R \ge 2r$.

6) In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{AH\cdot AI}{\sqrt{r_b+r_c}}\right)^2 \ge 2R^2r$$

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Solution. Using Lemma, we get:

$$\frac{R[s^4-2s^2r(4R+5r)+r^2(4R+r)^2]}{s^2} = R\left[s^2-2r(4R+5r)+\frac{r^2(4R+r)^2}{s^2}\right] \ge$$

$$\ge R\left[16Rr-5r^2-2r(4R+r)+\frac{r^2(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}\right] =$$

$$= R\left[s^2-2r(4R+5r)+\frac{r^2(4R+r)^2}{s^2}\right] = 8R^2r-15Rr^2+2r^2(2R-r) =$$

$$=r(8R^2-11Rr-2r^2)\overset{Euler}{\geq} r\cdot 2R^2=2R^2r$$, which it follows from Gerretsen:

$$\frac{r(4R+r)^2}{R+r}\leq 16Rr-5r^2\leq 4R^2+4Rr+3r^2$$

Equality holds if and only if triangle is equilateral.

7) In ΔABC the following relationship holds:

$$2R^2r\leq\sum_{cyc}\left(\frac{AH\cdot AI}{\sqrt{r_b+r_c}}\right)^2\leq R(4R^2-3Rr-6r^2)$$

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Solution. **Lemma. 8) In ΔABC the following relationship holds:**

$$\sum_{cyc}\left(\frac{AH\cdot AI}{\sqrt{r_b+r_c}}\right)^2=\frac{R[s^4-2s^2r(4R+5r)+r^2(\$r=R+r)^2]}{s^2}$$

Proof. Using $AH=4R^2cos^2A=4R^2-a^2, AI=\frac{r^2}{sin^2\frac{A}{2}}=bc-4Rr$,

$r_b+r_c=\frac{F}{s-b}+\frac{F}{s-c}=4Rcos\frac{A}{2}$, it follows that:

$$\sum_{cyc}\left(\frac{AH\cdot AI}{\sqrt{r_b+r_c}}\right)^2=\sum_{cyc}\frac{AH^2\cdot AI^2}{r_b+r_c}=\sum_{cyc}\frac{(4R^2-a^2)(bc-4Rr)}{4Rcos^2\frac{A}{2}}=$$

$$=\frac{1}{4R}\sum_{cyc}\frac{(4R^2-a^2)(bc-4Rr)}{cos^2\frac{A}{2}}=\frac{1}{4R}\sum_{cyc}\frac{(4R^2-a^2)(bc-4Rr)}{\frac{s(s-a)}{bc}}=$$

$$=\frac{1}{4Rs}\cdot\frac{\sum bc(s-b)(s-c)(4R^2-a^2)(bc-4Rr)}{\prod(s-a)}=$$

$$=\frac{1}{4Rs}\cdot\frac{4R^2r^2[s^4-2s^2r(4R+5r)+r^2(4R+r)^2]}{r^2s}=$$

$$=\frac{R[s^4-2s^2r(4R+5r)+r^2(4R+r)^2]}{s^2}$$, which follows from

$$\sum_{cyc}b^2c^2(s-b)(s-c)=r^2[s^2(s^2+2r^2-4Rr)+r(4R+r)^3]$$

$$\sum_{cyc}bc(s-b)(s-c)=r^2[s^2+(4R+r)^2]$$

$$\sum_{cyc}a(s-b)(s-c)=2sr(2R-r)$$

$$\sum_{cyc}(s-b)(s-c)=r(4R+r)$$

For RHS, using Lemma, inequality can be written as:

$$\frac{R[s^4-2s^2r(4R+5r)+r^2(4R+r)^2]}{s^2}=R\left[s^2-2r(4R+5r)+\frac{r^2(4R+r)^2}{s^2}\right]\le$$

$$\le R\left[4R^2+4Rr+3r^2-2r(4R+5r)+\frac{r^2(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$=R[4R^2-4Rr-7r^2+r(R+r)]=R(4R^2-3Rr-6r^2)$, which follows from Gerretsen inequality: $\frac{r(4R+r)^2}{R+r}\le 16Rr-5r^2\le 4R^2+4Rr+3r^2$.

Equality holds if and only if triangle is equilateral.

For LHS, using Lemma, inequality can be written as:

$$\frac{R[s^4-2s^2r(4R+5r)+r^2(4R+r)^2]}{s^2}=R\left[s^2-2r(4R+5r)+\frac{r^2(4R+r)^2}{s^2}\right]\ge$$

$$\ge R\left[16Rr-5r^2-2r(4R+r)+\frac{r^2(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}\right]=$$

$$=R\left[s^2-2r(4R+5r)+\frac{r^2(4R+r)^2}{s^2}\right]=8R^2r-15Rr^2+2r^2(2R-r)=$$

$=r(8R^2-11Rr-2r^2)\overset{Euler}{\ge} r\cdot 2R^2=2R^2r$, which it follows from Gerretsen:

$$\frac{r(4R+r)^2}{R+r}\le 16Rr-5r^2\le 4R^2+4Rr+3r^2$$

Equality holds if and only if triangle is equilateral.

9) In ΔABC the following relationship holds:

$$\mathbf{2R^2r\le r(8R^2-11Rr-2r^2)\le\sum_{cyc}\left(\frac{AH\cdot AI}{\sqrt{r_b+r_c}}\right)^2\le R(4R^2-3Rr-6r^2)}$$

$$\mathbf{\le 4R(R^2-3r^2)}$$

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Solution. See up these inequalities. Equality holds if and only if triangle is equilateral.

REFERENCE:

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