A SIMPLE PROOF FOR SCHWEITZER'S INEQUALITIES AND APPLICATIONS

DANIEL SITARU, CLAUDIA NĂNUŢI - ROMANIA

Abstract. In this paper it is proved Schweitzer's inequality in general form and particular cases. Also we give a few applications.

Schweitzer's inequality:
If $0 < \alpha \leq a_1, a_2, \ldots, a_n \leq \beta; n \in \mathbb{N}; n \geq 2$ then:

$$(a_1 + a_2 + \ldots + a_n)\Big(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\Big) \leq \frac{(\alpha+\beta)n^2}{4\alpha\beta}$$

Proof.
Lemma 1
If $x, y \in \mathbb{R}$ then: $4xy \leq (x+y)^2$

Proof.

$$(x-y)^2 \geq 0 \Rightarrow x^2 - 2xy + y^2 \geq 0 \Rightarrow x^2 + 2xy + y^2 \geq 4xy$$
$$\Rightarrow (x+y)^2 \geq 4xy \Rightarrow 4xy \leq (x+y)^2$$

□

Lemma 2
If $0 < \alpha \leq x \leq \beta$ then:

$$\frac{x}{\sqrt{\alpha\beta}} + \frac{\sqrt{\alpha\beta}}{x} \leq \frac{\alpha+\beta}{\sqrt{\alpha\beta}}$$

Proof.

$$\alpha \leq x \leq \beta \Rightarrow x - \alpha \geq 0; y - \beta \leq 0 \Rightarrow$$
$$(x-\alpha)(y-\beta) \leq 0 \Rightarrow x^2 - (\alpha+\beta)x + \alpha\beta \leq 0$$
$$x^2 + \alpha\beta \leq (\alpha+\beta)x \Rightarrow x + \frac{\alpha\beta}{x} \leq \alpha + \beta \Rightarrow$$
$$\Rightarrow \frac{x}{\sqrt{\alpha\beta}} + \frac{\sqrt{\alpha\beta}}{x} \leq \frac{\alpha+\beta}{\sqrt{\alpha\beta}}$$

□

Main proof:

$$4(a_1 + a_2 + \ldots + a_n)\Big(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\Big) =$$
$$= 4 \cdot \frac{1}{\sqrt{\alpha\beta}} \cdot \sqrt{\alpha\beta}(a_1 + a_2 + \ldots + a_n)\Big(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\Big) =$$
$$= 4\Big(\frac{a_1}{\sqrt{\alpha\beta}} + \frac{a_2}{\sqrt{\alpha\beta}} + \ldots + \frac{a_n}{\sqrt{\alpha\beta}}\Big)\Big(\frac{\sqrt{\alpha\beta}}{a_1} + \frac{\sqrt{\alpha\beta}}{a_2} + \ldots + \frac{\sqrt{\alpha\beta}}{a_n}\Big) \leq$$

$$\overset{\text{Lemma 1}}{\leq}\Big(\frac{a_1}{\sqrt{\alpha\beta}}+\frac{\sqrt{\alpha\beta}}{a_1}+\frac{a_2}{\sqrt{\alpha\beta}}+\frac{\sqrt{\alpha\beta}}{a_2}+\ldots+\frac{a_n}{\sqrt{\alpha\beta}}+\frac{\sqrt{\alpha\beta}}{a_n}\Big)^2\leq$$

$$\overset{\text{Lemma 2}}{\leq}\Big(\underbrace{\frac{\alpha+\beta}{\sqrt{\alpha\beta}}+\frac{\alpha+\beta}{\sqrt{\alpha\beta}}+\ldots+\frac{\alpha+\beta}{\sqrt{\alpha\beta}}}_{\text{for } n \text{ times}}\Big)^2=\Big(\frac{n(\alpha+\beta)}{\sqrt{\alpha\beta}}\Big)^2=\frac{(\alpha+\beta)^2n^2}{\alpha\beta}$$

Case $n=2$:
If $0<\alpha\leq a,b\leq\beta$ then:

$$(a+b)\Big(\frac{1}{a}+\frac{1}{b}\Big)\leq\frac{(\alpha+\beta)^2}{\alpha\beta} \tag{1}$$

Case $n=3$:
If $0<\alpha\leq a,b,c\leq\beta$ then:

$$(a+b+c)\Big(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\Big)\leq\frac{9(\alpha+\beta)^2}{4\alpha\beta} \tag{2}$$

Case $n=4$:
If $0<\alpha\leq a,b,c,d\leq\beta$ then:

$$(a+b+c+d)\Big(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\Big)\leq\frac{4(\alpha+\beta)^2}{\alpha\beta} \tag{3}$$

Corollary 1:
If $0<\alpha\leq x,y\leq\beta$ then:

$$\Big(\sqrt{xy}+\frac{x+y}{2}\Big)\Big(\frac{1}{\sqrt{xy}}+\frac{2}{x+y}\Big)\leq\frac{(\alpha+\beta)^2}{\alpha\beta}$$

Proof. We take in (1):

$$a=\sqrt{xy};b=\frac{x+y}{2}$$

□

Corollary 2:
If $0<\alpha\leq x,y\leq\beta$ then:

$$\Big(\sqrt{xy}+\sqrt{\frac{x^2+y^2}{2}}\Big)\Big(\frac{1}{\sqrt{xy}}+\sqrt{\frac{2}{x^2+y^2}}\Big)\leq\frac{(\alpha+\beta)^2}{\alpha\beta}$$

Proof. We take in (1):

$$a=\sqrt{xy};b=\sqrt{\frac{x^2+y^2}{2}}$$

□

Corollary 3:
If $0<\alpha\leq x,y\leq\beta$ then:

$$\Big(\frac{2xy}{x+y}+\sqrt{xy}\Big)\Big(\frac{1}{\sqrt{xy}}+\frac{x+y}{2xy}\Big)\leq\frac{(\alpha+\beta)^2}{\alpha\beta}$$

We take in (1) : $a=\frac{2xy}{x+y};b=\sqrt{xy}$. □

Corollary 4:
If $0 < \alpha \leq x, y \leq \beta$ then:

$$\Big(\frac{2xy}{x+y} + \frac{x+y}{2}\Big)\Big(\frac{x+y}{2xy} + \frac{2}{x+y}\Big) \leq \frac{(\alpha+\beta)^2}{\alpha\beta}$$

We take in (1):

$$a = \frac{2xy}{x+y}; b = \frac{x+y}{2}$$

Corollary 5:
If $0 < \alpha \leq x, y \leq \beta$ then:

$$\Big(\frac{x+y}{2} + \sqrt{\frac{x^2+y^2}{2}}\Big)\Big(\frac{2}{x+y} + \sqrt{\frac{2}{x^2+y^2}}\Big) \leq \frac{(\alpha+\beta)^2}{\alpha\beta}$$

We take in (1):

$$a = \frac{x+y}{2}; b = \sqrt{\frac{x^2+y^2}{2}}$$

Corollary 6:
If $0 < \alpha \leq x, y, z \leq \beta$ then:

$$\Big(\frac{3xyz}{xy+yz+zx} + \sqrt[3]{xyz} + \frac{x+y+z}{3}\Big)\Big(\frac{xy+yz+zx}{3xyz} + \frac{1}{\sqrt[3]{xyz}} + \frac{3}{x+y+z}\Big) \leq$$

$$\leq \frac{9(\alpha+\beta)^2}{4\alpha\beta}$$

Proof. We take in (2):

$$a = \frac{3xyz}{xy+yz+zx}; b = \sqrt[3]{xyz}; c = \frac{x+y+z}{3}$$

□

Corollary 7:
If $0 < \alpha \leq x, y, z \leq \beta$ then:

$$\Big(\frac{3xyz}{xy+yz+zx} + \sqrt[3]{xyz} + \sqrt{\frac{x^2+y^2+z^2}{3}}\Big)\Big(\frac{xy+yz+zx}{3xyz} + \frac{1}{\sqrt[3]{xyz}} + \sqrt{\frac{3}{x^2+y^2+z^2}}\Big) \leq$$

$$\leq \frac{9(\alpha+\beta)^2}{4\alpha\beta}$$

Proof. We take in (2):

$$a = \frac{3xyz}{xy+yz+zx}; b = \sqrt[3]{xyz}; c = \sqrt{\frac{x^2+y^2+z^2}{3}}$$

□

Corollary 8:
If $0 < \alpha \leq x, y, z \leq \beta$ then:

$$\Big(\sqrt[3]{xyz} + \frac{x+y+z}{3} + \sqrt{\frac{x^2+y^2+z^2}{3}}\Big)\Big(\frac{1}{\sqrt[3]{xyz}} + \frac{3}{x+y+z} + \sqrt{\frac{3}{x^2+y^2+z^2}}\Big) \leq$$

$$\leq \frac{9(\alpha+\beta)^2}{4\alpha\beta}$$

Proof. We take in (2):

$$a = \sqrt[3]{xyz}; b = \frac{x+y+z}{3}; c = \sqrt{\frac{x^2+y^2+z^2}{3}}$$

□

Corollary 9:
If $0 < \alpha \leq x, y, z, t \leq \beta$ then:

$$(m_h + m_g + m_a + m_q)\Big(\frac{1}{m_h} + \frac{1}{m_g} + \frac{1}{m_a} + \frac{1}{m_q}\Big) \leq \frac{4(\alpha+\beta)^2}{\alpha\beta}$$

$$m_h = \frac{4xyzt}{xyz + xyt + xzt + yzt}; m_g = \sqrt[4]{xyzt}$$

$$m_a = \frac{x+y+z+t}{4}; m_q = \sqrt{\frac{x^2+y^2+z^2+t^2}{4}}$$

Proof. We take in (3):

$$a = m_h; b = m_g; c = m_a; d = m_q$$

□

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania
Email address: dansitaru63@yahoo.com