$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}arg(2k^2+i),\quad i^2=-1.$$

Proposed by Marian Ursărescu-Romania

Solution by proposer

$$x_n=2n^2+i\to tan\,(argx_n)=\frac{1}{2n^2}\to argx_n=tan^{-1}\left(\frac{1}{2n^2}\right)\to$$

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}arg(2k^2+i)=\lim_{n\to\infty}\sum_{k=1}^{n}tan^{-1}\left(\frac{1}{2k^2}\right)=$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\left(tan^{-1}\left(\frac{k}{k+1}\right)-tan^{-1}\left(\frac{k-1}{k}\right)\right)=\lim_{n\to\infty}tan^{-1}\left(\frac{n}{n+1}\right)=tan^{-1}1=\frac{\pi}{4}$$

$$\because\frac{1}{2k^2}=\frac{\frac{k}{k+1}-\frac{k-1}{k}}{1+\frac{k(k-1)}{k(k+1)}}$$

1267. $p\in\mathbb{N}^*,p\geq2,0<x_1<1,x_{n+1}=x_n(1-x_n)^p,y_1>0,y_{n+1}=y_n+\frac{1}{y_n^p},n\geq1$

Find:

$$\Omega=\lim_{p\to\infty}\left(\lim_{n\to\infty}x_n\cdot y_n^{p+1}\right)^p$$

Proposed by Marian Ursărescu-Romania

Solution by proposer

$$x_n\in(0,1),(x_n)_{n\geq1}\searrow,\text{ then }\lim_{n\to\infty}x_n=0.$$

$$y_n>0,(y_n)_{n\geq1}\nearrow,\text{ then }\lim_{n\to\infty}y_n=+\infty$$

$$\lim_{n\to\infty}x_n\cdot y_n^{p+1}=\lim_{n\to\infty}nx_n\cdot\frac{y_n^{p+1}}{n};(1)$$

$$\lim_{n\to\infty}nx_n=\lim_{n\to\infty}\frac{n}{\frac{1}{x_n}}\overset{LCS}{=}\lim_{n\to\infty}\frac{n+1-n}{\frac{1}{x_{n+1}}-\frac{1}{x_n}}=\lim_{n\to\infty}\frac{x_nx_{n+1}}{x_n-x_{n+1}}=\lim_{n\to\infty}\frac{x_n^2(1-x_n)^p}{x_n-x_n(1-x_n)^p}=$$

$$=\lim_{n\to\infty}\frac{x_n(1-x_n)^p}{1-(1-x_n)^p}=\lim_{n\to\infty}\frac{x_n(1-x_n)^p}{x_n(1+(1-x_n)+\cdots+(1-x_n)^{p-1})}=\frac{1}{p};(2)$$

$$\lim_{n\to\infty}\frac{y_n^{p+1}}{n}\overset{LCS}{=}\lim_{n\to\infty}\frac{y_{n+1}^{p+1}-y_n^{p+1}}{n+1-n}=\lim_{n\to\infty}\left[\left(y_n+\frac{1}{y_n^p}\right)^{p+1}-y_n^{p+1}\right]=$$

$$=\lim_{n\to\infty}\left(y_n^{p+1}+\binom{p+1}{1}y_n^p\cdot\frac{1}{y_n^p}+\dots+\binom{p+1}{p+1}\frac{1}{(y_n^p)^{p+1}}-y_n^{p+1}\right)=p+1;(3)$$

From (1),(2),(3), it follows that:

$$\lim_{n\to\infty}x_n\cdot y_n^{p+1}=\frac{p+1}{p}\to\Omega=\lim_{p\to\infty}\left(\lim_{n\to\infty}x_n\cdot y_n^{p+1}\right)^p=\lim_{p\to\infty}\left(\frac{p+1}{p}\right)^p=e$$

1268. Find:

$$\Omega=\lim_{n\to\infty}\left((n-1)\cdot\frac{1}{n}+(n-2)\cdot\left(\frac{1}{n}+\frac{1}{n-1}\right)+(n-3)\cdot\left(\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}\right)+\dots+1\cdot\left(\frac{1}{n}+\frac{1}{n-1}+\dots+\frac{1}{2}\right)\right)tan\frac{1}{n^2}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ruxandra Daniela Tonilă-Romania

$$(n-1)\cdot\frac{1}{n}+(n-2)\cdot\left(\frac{1}{n}+\frac{1}{n-1}\right)+(n-3)\cdot\left(\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}\right)+\dots+1\cdot\left(\frac{1}{n}+\frac{1}{n-1}+\dots+\frac{1}{2}\right)=$$

$$=\frac{1}{n}(n-1+n-2+n-3+\dots+1)+\frac{1}{n-1}(n-2+n-3+\dots+1)+$$

$$+\frac{1}{n-2}(n-3+n-4+\dots+1)+\dots+\frac{1}{2}\cdot 1=$$

$$=\frac{1}{n}\cdot\frac{(n-1)n}{2}+\frac{1}{n-1}\cdot\frac{(n-2)(n-1)}{2}+\frac{1}{n-2}\cdot\frac{(n-3)(n-4)}{2}+\dots+\frac{1}{2}\cdot 1=$$

$$=\frac{1}{2}\cdot\frac{n(n-1)}{2}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\frac{n(n-1)}{2}\cdot tan\frac{1}{n^2}=\lim_{n\to\infty}\frac{n-1}{4}\cdot\frac{tan\frac{1}{n^2}}{\frac{1}{n^2}}=\frac{1}{4}$$

Solution 2 by Ay Men-Algerie

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n-1}(n-k)\cdot\sum_{m=0}^{k-1}\frac{1}{n-m}\right)\cdot tan\frac{1}{n^2}=$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n-1}(n-k)\left(\Psi(-n)-\Psi(k-n)\right)\cdot tan\frac{1}{n^2}=$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n-1}(n-k)\big(logn - log(n-k)\big)\cdot tan\frac{1}{n^2} =$$

$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n-1}(n-k)\big(logn - log(n-k)\big)\right)\cdot tan\frac{1}{n^2} =$$

$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n-1}(n-k)\cdot logn - \sum_{k=1}^{n-1}(n-k)\cdot log(n-k)\right)\cdot tan\frac{1}{n^2} =$$

$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n-1}(n-k)\cdot logn - \sum_{k=1}^{n-1}klogk\right)\cdot tan\frac{1}{n^2} =$$

$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n-1}(n-k)\cdot logn - \sum_{k=1}^{n-1}k\left(logn + log\frac{k}{n}\right)\right)\cdot tan\frac{1}{n^2} =$$

$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n-1}(n-k)\cdot logn - \sum_{k=1}^{n-1}klogn - \sum_{k=1}^{n-1}klog\frac{k}{n}\right)\cdot\left(\frac{1}{n^2} + o\left(\frac{1}{n^6}\right)\right) =$$

$$= -\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1}\frac{k}{n}log\left(\frac{k}{n}\right) = -\int_0^1 xlogx\,dx = \frac{1}{4}$$

Therefore,

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n-1}(n-k)\sum_{m=0}^{k-1}\frac{1}{n-m}\right)\cdot tan\frac{1}{n^2} = \frac{1}{4}$$

1269. Find:

$$\Omega = \lim_{n\to\infty} n^2\left(e^{H_n-\gamma} - n\right)\left(e^{\frac{3}{n+5}} - e^{\frac{3}{n+7}}\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Khaled Abd Imouti-Damascus-Syria

$$u_n = n^2\left(e^{H_n-\gamma} - n\right)\left(e^{\frac{3}{n+5}} - e^{\frac{3}{n+7}}\right) =$$

$$= n^2\left(ne^{\frac{1}{2n}+\frac{\epsilon}{n^2}} - n\right)\left(e^{\frac{3}{n+5}} - e^{\frac{3}{n+7}}\right) = -n^3\left(e^{\frac{n+2\epsilon}{2n^2}} - 1\right)\cdot e^{\frac{3}{n+5}}\left(e^{\frac{6}{(n+5)(n+7)}} - 1\right) =$$

$$=-n^3\cdot\frac{n+2\epsilon}{2n^2}\cdot\frac{-6}{(n+5)(n+7)}\overbrace{\left(\frac{e^{\frac{n+2\epsilon}{2n^2}}-1}{\frac{n+2\epsilon}{2n^2}}\right)}^{v_n}\cdot e^{\frac{3}{n+5}}\overbrace{\left(\frac{e^{\frac{-6}{(n+5)(n+7)}}-1}{-\frac{6}{(n+5)(n+7)}}\right)}^{t_n}$$

$$u_n=\frac{6n^3(n+2\epsilon)}{2n^2(n^2+12n+35)}\cdot e^{\frac{3}{n+5}}\cdot v_n\cdot t_n\to\lim_{n\to\infty}u_n=3$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$\because H_n\sim\log n+\gamma+\frac{1}{2n}-\frac{1}{12n^2}\left(1+o\left(\frac{1}{n^2}\right)\right)$$

$$u_n=n^2\left(e^{H_n-\gamma}-n\right)\left(e^{\frac{3}{n+5}}-e^{\frac{3}{n+7}}\right)$$

$$H_n-\gamma\cong\log n+\frac{1}{2n}-\frac{1}{12n^2}$$

$$e^{H_n-\gamma}-n\cong ne^{\frac{1}{2n}-\frac{1}{12n^2}}-n=n^2\left(e^{\frac{6n-1}{12n^2}}-1\right)$$

$$u_n=n^2\left(ne^{\frac{1}{2n}+\frac{\epsilon}{n^2}}-n\right)\left(e^{\frac{3}{n+5}}-e^{\frac{3}{n+7}}\right)=-n^3\left(e^{\frac{n+2\epsilon}{2n^2}}-1\right)\cdot e^{\frac{3}{n+5}}\left(e^{\frac{6}{(n+5)(n+7)}}-1\right)=$$

$$=-n^3\cdot\frac{6n-1}{12n^2}\cdot\frac{e^{\frac{6n-1}{12n^2}}-1}{\frac{6n-1}{12n^2}}\cdot e^{\frac{3}{n+5}}\cdot\frac{e^{-\frac{6}{(n+5)(n+7)}}-1}{-\frac{6}{(n+5)(n+7)}}\cdot\frac{-6}{(n+5)(n+7)}$$

Therefore,

$$\Omega=\lim_{n\to\infty}n^2\left(e^{H_n-\gamma}-n\right)\left(e^{\frac{3}{n+5}}-e^{\frac{3}{n+7}}\right)=3$$

Solution 3 by Mikael Bernardo-Mozambique

$$\Omega=\lim_{n\to\infty}n^2\left(e^{H_n-\gamma}-n\right)\left(e^{\frac{3}{n+5}}-e^{\frac{3}{n+7}}\right)=$$

$$=\lim_{n\to\infty}n^2\left(e^{H_n-\gamma}-e^{\log n}\right)e^{\frac{3}{n+7}}\left(e^{\frac{3}{n+5}-\frac{3}{n+7}}-1\right)=$$

$$=\lim_{n\to\infty}n^2e^{\log n}\left(e^{(H_n-\log n)-\gamma}-1\right)e^{\frac{3}{n+7}}\left(e^{\frac{6}{(n+5)(n+7)}}-1\right)=$$

$$=\lim_{n\to\infty}\frac{e^{(H_n-logn)-\gamma}-1}{(H_n-logn)-\gamma}\cdot\left((H_n-logn)-\gamma\right)\cdot\lim_{n\to\infty}e^{\frac{3}{n+7}}\cdot\frac{e^{\frac{6}{(n+5)(n+7)}}-1}{\frac{6}{(n+5)(n+7)}}\cdot\frac{1}{\frac{(n+5)(n+7)}{6}}\cdot n^3$$

$$=\lim_{n\to\infty}n(H_n-logn-\gamma)=3$$

$$\because H_n\sim logn+\gamma+\frac{1}{2n}-\frac{1}{12n^2}\left(1+o\left(\frac{1}{n^2}\right)\right)$$

$$H_n-logn-\gamma=\frac{1}{2n}-\frac{1}{12n^2}\left(1+o\left(\frac{1}{n^2}\right)\right)\to\lim_{n\to\infty}\left(\frac{1}{2n}-\frac{1}{12n^2}\left(1+o\left(\frac{1}{n^2}\right)\right)\right)n=\frac{1}{2}$$

1270. Find:

$$\Omega=\lim_{n\to\infty}\left((1\cdot n+3\cdot(n-1)+5\cdot(n-2)+\cdots+(2n-1)\cdot 1)\cdot sin\frac{1}{n^3})\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$\Omega=\lim_{n\to\infty}\left((1\cdot n+3\cdot(n-1)+5\cdot(n-2)+\cdots+(2n-1)\cdot 1)\cdot sin\frac{1}{n^3})\right)=$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}(2k-1)(n-k+1)\cdot sin\frac{1}{n^3}=\lim_{n\to\infty}(2kn-2k^2+2k-n-n+k-1)sin\frac{1}{n^3}$$

$$=\lim_{n\to\infty}\frac{2n^3+3n^2+n}{6}\cdot sin\frac{1}{n^3}=\lim_{n\to\infty}\frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}\cdot\frac{sin\frac{1}{n^3}}{\frac{1}{n^3}}=\frac{1}{3}$$

Solution 2 by George Florin Şerban-Romania

$$S=(1\cdot n+3\cdot(n-1)+5\cdot(n-2)+\cdots+(2n-1)\cdot 1)=\sum_{k=1}^{n}(2k-1)[n-(k-1)]$$

$$=n\sum_{k=1}^{n}(2k-1)-\sum_{k=1}^{n}(2k^2-3k+1)=\frac{n(n+1)(2n+1)}{6}$$

$$\Omega=\lim_{n\to\infty}\left((1\cdot n+3\cdot(n-1)+5\cdot(n-2)+\cdots+(2n-1)\cdot 1)\cdot sin\frac{1}{n^3})\right)=$$

$$= \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6} \cdot \frac{\sin\frac{1}{n^3}}{\frac{1}{n^3}} \cdot \frac{1}{n^3} = \frac{1}{3}$$

Solution 3 by Khaled Abd Imouti-Damascus-Syria

$$u_n = (1\cdot n + 3\cdot(n-1) + 5\cdot(n-2) + \dots + (2n-1)\cdot 1) = \sum_{k=1}^{n} (2k-1)[n-(k-1)]$$

$$= 2n\sum_{k=1}^{n} k - 2\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k - n^2 - n = \frac{n(n+1)(2n+1)}{6}$$

$$\Omega = \lim_{n\to\infty}\left((1\cdot n + 3\cdot(n-1) + 5\cdot(n-2) + \dots + (2n-1)\cdot 1)\cdot \sin\frac{1}{n^3})\right) =$$

$$= \lim_{n\to\infty} u_n \cdot \sin\frac{1}{n^3} = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6} \cdot \frac{\sin\frac{1}{n^3}}{\frac{1}{n^3}} \cdot \frac{1}{n^3} = \frac{1}{3}$$

Solution 4 by Surjeet Singhania-India

$$\Omega = \lim_{n\to\infty}\left((1\cdot n + 3\cdot(n-1) + 5\cdot(n-2) + \dots + (2n-1)\cdot 1)\cdot \sin\frac{1}{n^3})\right) =$$

$$= \lim_{n\to\infty} \sin\frac{1}{n^3} \cdot \sum_{k=1}^{n} (2k-1)(n-k) =$$

$$= \lim_{n\to\infty} \sin\frac{1}{n^3} \cdot \left\{ n^2(n+1) - \frac{n(n+1)(2n+1)}{3} - n^2 + \frac{n(n+1)}{2} \right\} \sim$$

$$\sim \lim_{n\to\infty} \left\{\frac{1}{n^3} - o\left(\frac{1}{n^9}\right)\right\}\left\{n^3 - \frac{2}{3}n^3 - n^2 - \frac{1}{2}n^2\right\} = \frac{1}{3}$$

1271. **Find:**

$$\Omega = \lim_{x\to 0}\left(\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k\binom{n}{k}}{n2^n + k} \cdot \prod_{k=0}^{n} \cos(2^{k-n}x)\right)$$

Proposed by Florică Anastase-Romania

Solution by proposer

$$P_n = cosa \cdot cos2a \cdot \ldots \cdot cos2^n a = \prod_{k=1}^{n} cos2^k a\, , a \neq \frac{(2k+1)\pi}{2}$$

From $cosa = \frac{sin2a}{2sina}, \ldots, cos2^n a = \frac{sin2^{n+1}a}{2sin2^n a}$, we get:

$$P_n = \prod_{k=1}^{n} cos2^k a = \frac{sin2^{n+1}a}{2^{n+1}sina}, a \neq k\pi$$

For $a = \frac{x}{2^n}$, we get:

$$\lim_{n\to\infty} \prod_{k=0}^{n} cos(2^{k-n}x) = \frac{sin2x}{2x}$$

$$\because \sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

$$\sum_{k=1}^{n} \frac{k\binom{n}{k}}{n2^n + k} \geq \sum_{k=1}^{n} \frac{k\binom{n}{k}}{n2^n + n} = \frac{1}{n2^n + n}\sum_{k=1}^{n} k\binom{n}{k} = \frac{n2^{n-1}}{n2^n + n} = \frac{2^{n-1}}{2^n + 1}$$

$$\sum_{k=1}^{n} \frac{k\binom{n}{k}}{n2^n + k} \leq \sum_{k=1}^{n} \frac{k\binom{n}{k}}{n2^n + 1} = \frac{1}{n2^n + 1}\sum_{k=1}^{n} k\binom{n}{k} = \frac{n2^{n-1}}{n2^n + 1}$$

Thus,

$$\frac{2^{n-1}}{2^n + 1} \leq \sum_{k=1}^{n} \frac{k\binom{n}{k}}{n2^n + k} \leq \frac{n2^{n-1}}{n2^n + 1} \Rightarrow \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k\binom{n}{k}}{n2^n + k} = \frac{1}{2}$$

Thus,

$$\Omega = \lim_{x\to 0}\left(\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k\binom{n}{k}}{n2^n + k} \cdot \prod_{k=0}^{n} cos(2^{k-n}x)\right) = \frac{1}{2}\lim_{x\to 0}\frac{sin2x}{2x} = \frac{1}{2}$$

1272.

$\Omega(n) = \sum_{k=0}^{n} \int_0^1 x^k (tan^{-1}x)^{k+1}\, dx$. Find:

$$\lim_{n\to\infty} \frac{\Omega(n)}{n}$$

Proposed by Costel Florea-Romania

Solution 1 by Adrian Popa-Romania

$$\Omega(n)=\sum_{k=0}^{n}\int_0^1 x^k(tan^{-1}x)^{k+1}\,dx=\int_0^1 tan^{-1}x\cdot\sum_{k=0}^{n}(xtan^{-1}x)^k\,dx=$$

$$=\int_0^1 tan^{-1}x\cdot\frac{(xtan^{-1}x)^{n+1}-1}{xtan^{-1}x-1}dx$$

$$x\in[0,1]\Rightarrow tan^{-1}x\in\left[0,\frac{\pi}{4}\right]\Rightarrow x\cdot tan^{-1}x<1\Rightarrow(x\cdot tan^{-1}x)^n\to 0, if\ n\to\infty$$

$$\lim_{n\to\infty}\frac{\Omega(n)}{n}=\lim_{n\to\infty}\frac{\int_0^1\frac{tan^{-1}x}{1-x\cdot tan^{-1}x}dx}{n}$$

Let $f(x)=\frac{tan^{-1}x}{1-x\cdot tan^{-1}x}, f'(x)=\frac{\frac{1}{1+x^2}+(tan^{-1}x)^2}{(1+x\cdot tan^{-1}x)^2}>0\Rightarrow f\uparrow[0,1], f(0)=0, f(1)=\frac{\pi}{4-\pi}$

Therefore,

$$\lim_{n\to\infty}\frac{\Omega(n)}{n}=\lim_{n\to\infty}\frac{\int_0^1\frac{tan^{-1}x}{1-x\cdot tan^{-1}x}dx}{n}=0$$

Solution 2 by Mikael Bernardo-Mozambique

$$\Omega(n)=\sum_{k=0}^{n}\int_0^1 x^k(tan^{-1}x)^{k+1}\,dx=\int_0^1 tan^{-1}xdx+\int_0^1 tan^{-1}x\sum_{k=1}^{n}(xtan^{-1}x)^k\,dx\overset{IBP}{\Longrightarrow}$$

$$0<x\cdot tan^{-1}x<1\Rightarrow\sum_{k=1}^{n}(xtan^{-1}x)^k=\frac{x\cdot tan^{-1}x-(x\cdot tan^{-1}x)^{n+1}}{1-x\cdot tan^{-1}x}$$

$$\Omega(n)=[x\cdot tan^{-1}x]_0^1-\int_0^1\frac{x}{1+x^2}dx+\int_0^1 tan^{-1}x\left(\frac{x\cdot tan^{-1}x(1-x\cdot tan^{-1}x)^n}{1-x\cdot tan^{-1}x}\right)dx$$

$$=\frac{\pi}{4}-\frac{1}{2}\int_0^1\frac{d(1+x^2)}{1+x^2}dx+\int_0^1\frac{x\cdot(tan^{-1}x)^2}{1-x\cdot tan^{-1}x}dx-\int_0^1\frac{tan^{-1}x\cdot(x\cdot tan^{-1}x)^{n+1}}{1-x\cdot tan^{-1}x}dx=$$

$$=\frac{\pi}{4}-\frac{1}{2}log2+\int_0^1\frac{x\cdot(tan^{-1}x)^2}{1-x\cdot tan^{-1}x}dx-\int_0^1\frac{tan^{-1}x\cdot(x\cdot tan^{-1}x)^{n+1}}{1-x\cdot tan^{-1}x}dx$$

Now,

$$\lim_{n\to\infty}\frac{\Omega(n)}{n}=\lim_{n\to\infty}\frac{\pi}{4n}-\lim_{n\to\infty}\frac{1}{2n}\log 2+\lim_{n\to\infty}\frac{1}{n}\int_0^1\frac{x\cdot(\tan^{-1}x)^2}{1-x\cdot\tan^{-1}x}dx$$

$$-\lim_{n\to\infty}\int_0^1\frac{\tan^{-1}x\cdot(x\cdot\tan^{-1}x)^{n+1}}{1-x\cdot\tan^{-1}x}dx=$$

$$=-\lim_{n\to\infty}\int_0^1\frac{\tan^{-1}x\cdot(x\cdot\tan^{-1}x)^{n+1}}{1-x\cdot\tan^{-1}x}dx$$

$$f(x)=\frac{\tan^{-1}x\cdot(x\cdot\tan^{-1}x)^{n+1}}{1-x\cdot\tan^{-1}x}$$

$$f'(x)=$$

$$=\frac{\left(\frac{(x\tan^{-1}x)^{n+1}}{1+x^2}+\tan^{-1}x\left((n+1)(x\tan^{-1}x)^n\left(\tan^{-1}x+\frac{x}{1+x^2}\right)\right)\right)(1-x\tan^{-1}x)-(\tan^{-1}x\cdot(x\tan^{-1}x)^{n+1}\left(-\tan^{-1}x-\frac{x}{1+x^2}\right)}{(1-x\tan^{-1}x)^2}$$

$$f'(0)=0$$

$$f'(1)=\frac{\left(\frac{\left(\frac{\pi}{4}\right)^{n+1}}{2}+\frac{\pi}{4}(n+1)\left(\frac{\pi}{4}\right)^n\left(\frac{\pi}{2}+\frac{1}{2}\right)\left(1-\frac{\pi}{4}\right)+\frac{\pi}{4}\left(\frac{\pi}{4}\right)^{n+1}\left(\frac{\pi}{4}+\frac{1}{2}\right)\right)}{\left(1-\frac{\pi}{4}\right)^2}$$

Since $0<a<1\Rightarrow\lim_{n\to\infty}\frac{\pi}{8}\left(\frac{\pi}{4}\right)^n=0,\ \lim_{n\to\infty}(n+1)\left(\frac{\pi}{4}\right)^n=0,\ \lim_{n\to\infty}\left(\frac{\pi}{4}\right)^{n+1}=0$. Thus,

$$\lim_{n\to\infty}\frac{\Omega(n)}{n}=0.$$

1273. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n-1}(n-k)\int_{\frac{k}{n}}^{\frac{k+1}{n}}\frac{\log(1+x)}{(1-x)(1+x^2)}dx\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Remus Florin Stanca-Romania

$x\to\frac{\log(x+1)}{(1-x)(x^2+1)}$ is continuous as $1-x>0\Rightarrow\exists c_k\in\left[\frac{k}{n};\frac{k+1}{n}\right]$ such that:

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}}\frac{\log(1+x)}{(1-x)(1+x^2)}dx=\left(\frac{k+1}{n}-\frac{k}{n}\right)\frac{\log(1+c_k)}{(1-c_k)(1+c_k^2)}\to$$

$$l=\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\frac{log(1+c_k)}{(1-c_k)(1+c_k^2)}\right)=\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n-1}(1-c_k)\frac{log(1+c_k)}{(1-c_k)(1+c_k^2)}\right)=$$

$$=\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n-1}\frac{log(1+c_k)}{1+c_k^2}\right)$$

Let x_k such that $\lim_{n\to\infty}\|\Delta_n\|=0, \|\Delta_n\|=\max_{1\le k\le n}(x_{k+1}-x_k), a=\lim_{n\to\infty}x_1, b=\lim_{n\to\infty}x_n$

$c_k\in[x_n,x_{n+1}]$ and f –continuous function →

$$\lim_{n\to\infty}\sum_{k=1}^{n}(x_{k+1}-x_k)f(c_k)=\int_0^1\frac{log(x+1)}{x^2+1}dx\overset{x=\frac{1-t}{1+t}}{=}\int_0^1\frac{log2-log(1+x)}{x^2+1}dx=$$

$$=log2\int_0^1\frac{dx}{x^2+1}-\int_0^1\frac{log(x+1)}{x^2+1}dx\to$$

$$2\int_0^1\frac{log(x+1)}{x^2+1}dx=\frac{\pi}{4}log2\to\int_0^1\frac{log(x+1)}{x^2+1}dx=\frac{\pi}{8}log2$$

Therefore,

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n-1}(n-k)\int_{\frac{k}{n}}^{\frac{k+1}{n}}\frac{log(1+x)}{(1-x)(1+x^2)}dx\right)=\frac{\pi}{8}log2$$

Solution 2 by Mohammad Rostami-Kabul-Afghanistan

$f(x)=\frac{log(1+x)}{(1-x)(1+x^2)}, [a,b]=\left[\frac{k}{n},\frac{k+1}{n}\right], f$ –continuous in $[a,b]$ →

$\exists c\in[a,b]$ such that: $\int_a^b f(x)dx=(b-a)f(c)$

$$c\in\left[\frac{k}{n},\frac{k+1}{n}\right]\to c=\frac{k}{n}, I=\int_{\frac{k}{n}}^{\frac{k+1}{n}}\frac{log(1+x)}{(1-x)(1+x^2)}dx=\left(\frac{k+1}{n}-\frac{k}{n}\right)\frac{log(1+c)}{(1-c)(1+c^2)}=$$

$$=\frac{1}{n}\frac{log(1+c)}{(1-c)(1+c^2)}$$

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1}(n-k)\frac{1}{n}\frac{log(1+c)}{(1-c)(1+c^2)}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\frac{log\left(1+\frac{k}{n}\right)}{\left(1-\frac{k}{n}\right)\left[1+\left(\frac{k}{n}\right)^2\right]}$$

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{log\left(1+\frac{k}{n}\right)}{1+\left(\frac{k}{n}\right)^2}\cdot\frac{1}{n}=\int_0^1\frac{log(1+x)}{x^2+1}dx=\frac{\pi}{8}log2,$$

$$\because\Delta_x=\frac{b-a}{n}=\frac{1}{n},\lim_{n\to\infty}\sum_{k=1}^{n}f(x_k)\Delta_x=\int_a^b f(x)dx,x_k=a+k,\Delta_x=\frac{k}{n}$$

$$\Phi=\int_0^1\frac{log(1+x)}{1+x^2}dx\overset{x=tan\alpha}{=}\int_0^{\frac{\pi}{4}}\frac{log(1+tan\alpha)}{1+tan^2\alpha}(1+tan^2\alpha)d\alpha=$$

$$=\int_0^{\frac{\pi}{4}}log\left(\frac{sin\alpha+cos\alpha}{cos\alpha}\right)d\alpha=\int_0^{\frac{\pi}{4}}log\left[\sqrt{2}cos\left(\frac{\pi}{4}-\alpha\right)\right]d\alpha-\int_0^{\frac{\pi}{4}}log(cos\alpha)\,d\alpha=$$

$$=\int_0^{\frac{\pi}{4}}log\sqrt{2}\,d\alpha+\int_0^{\frac{\pi}{4}}log\left[cos\left(\frac{\pi}{4}-\alpha\right)\right]d\alpha-\int_0^{\frac{\pi}{4}}log(cos\alpha)\,d\alpha=\int_0^{\frac{\pi}{4}}log\sqrt{2}\,d\alpha$$

$$\Phi=\int_0^{\frac{\pi}{4}}log\sqrt{2}\,d\alpha=\frac{\pi}{8}log2$$

Therefore,

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n-1}(n-k)\int_{\frac{k}{n}}^{\frac{k+1}{n}}\frac{log(1+x)}{(1-x)(1+x^2)}dx\right)=\frac{\pi}{8}log2$$

Solution 3 by Nassim Nicholas Taleb-New York-USA

Let $f(x)=\frac{log(x+1)}{(1-x)(x^2+1)},F(x)=\int f(x)dx$

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}(n-k)\left(F\left(\frac{k+1}{n}\right)-F\left(\frac{k}{n}\right)\right)\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(f\left(\frac{i}{n}\right)-f\left(\frac{i-1}{n}\right)\right)=$$

$$=\int_0^1\frac{log(1+x)}{1+x^2}dx=\frac{\pi}{8}log2$$

1274. Find:

$$\Omega=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\zeta_n(3)}{n}$$

Proposed by Probal Chakraborty-India

Solution by proposer

$$\Omega=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\zeta_n(3)}{n}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{k=1}^{n}\frac{1}{k^3}=\sum_{k=1}^{n}\frac{1}{k^3}\sum_{n=k}^{\infty}\frac{(-1)^{n-1}}{n};(1)$$

$$\sum_{n=k}^{\infty}\frac{(-1)^{n-1}}{n}=\sum_{k=1}^{\infty}(-1)^{k-1}\int_0^1 x^{k-1}dx=\int_0^1\frac{(-x)^{k-1}}{1+x}dx;(2)$$

From (1),(2) we have:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\zeta_n(3)}{n}=\sum_{k=1}^{\infty}\frac{1}{k^3}\int_0^1\frac{(-x)^{n-1}}{1+x}dx=-\int_0^1\sum_{k=1}^{\infty}\frac{(-x)^k}{k^3}\cdot\frac{1}{x(x+1)}dx$$

$$=-\int_0^1\frac{Li_3(-x)}{x(x+1)}dx=\int_{-1}^0\frac{Li_3(x)}{x}dx+\int_{-1}^0\frac{Li_3(x)}{1-x}dx$$

$$-Li_4(-1)+Li_3(-1)\log 2-\int_{-1}^0 xLi_2'(x)\frac{Li_2(x)}{x}dx$$

Thus,

$$\Omega=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\zeta_n(3)}{n}=-Li_4(-1)+Li_3(-1)\log 2+\frac{1}{2}Li_2(-1)^2$$

$$=\frac{19\pi^4}{1440}-\frac{3}{4}\zeta(3)\log(2)$$

1275. Find:

$$\Omega=\lim_{n\to\infty}\left(\left(\sqrt[n]{\prod_{k=1}^{n}(k+n)}\right)^{-1}\cdot\sum_{k=1}^{n}\frac{1}{k}\cdot\sqrt[k]{\prod_{p=1}^{k}(k+p)}\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Remus Florin Stanca-Romania

$$\Omega=\lim_{n\to\infty}\left(\left(\sqrt[n]{\prod_{k=1}^{n}(k+n)}\right)^{-1}\cdot\sum_{k=1}^{n}\frac{1}{k}\cdot\sqrt[k]{\prod_{p=1}^{k}(k+p)}\right)=$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k}\cdot\sqrt[k]{\prod_{p=1}^{k}(k+p)}}{\frac{\sqrt[n]{\prod_{k=1}^{n}(k+n)}}{n}}; (1)$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{\prod_{k=1}^{n}(k+n)}}{n} = \lim_{n\to\infty}\sqrt[n]{\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)} = e^{\lim_{n\to\infty}\frac{1}{n}log\left(\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)\right)} = e^{\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}log\left(1+\frac{k}{n}\right)}; (2)$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}(x_{k+1}-x_k)f(\xi_k) = \int_a^b f(x)dx$$

f –continuous and $\lim_{n\to\infty}\|\Delta_n\| = 0, \|\Delta_n\| = \underset{1\le k\le n}{max}(x_{k+1}-x_k), a = \lim_{n\to\infty}x_1, b = \lim_{n\to\infty}x_n,$

$$\xi_k \in [x_k, x_{k+1}], x_k = a + \frac{b-a}{n}\cdot k$$

Let $f(x) = log(x+1), a = 0, b = 1 \Rightarrow x_k = \frac{k}{n} \Rightarrow \|\Delta_n\| = \frac{1}{n}, \lim_{n\to\infty}\|\Delta_n\| = 0$

$$\xi_k = \frac{k}{n} \in \left[\frac{k}{n}, \frac{k+1}{n}\right] \Rightarrow \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right) = \int_0^1 f(x)dx$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}log\left(1+\frac{k}{n}\right) = \int_0^1 log(1+x)\,dx = (x+1)(log(x+1)-1)|_0^1 = log4 - 1 \overset{(2)}{\Rightarrow}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{\prod_{k=1}^{n}(k+n)}}{n} = e^{log4-1} = \frac{4}{e}; (3)$$

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\frac{1}{k}\cdot\sqrt[k]{\prod_{p=1}^{k}(k+p)}}{n} \overset{L.C-S}{=} \lim_{n\to\infty}\frac{1}{n}\sqrt[n]{\prod_{p=1}^{n}(n+p)} \overset{(3)}{=} \frac{4}{e} \Rightarrow$$

$$\Omega = \lim_{n\to\infty}\left(\left(\sqrt[n]{\prod_{k=1}^{n}(k+n)}\right)^{-1}\cdot\sum_{k=1}^{n}\frac{1}{k}\cdot\sqrt[k]{\prod_{p=1}^{k}(k+p)}\right) = 1$$

Solution 2 by Adrian Popa-Romania

$$\Omega=\lim_{n\to\infty}\left(\left(\sqrt[n]{\prod_{k=1}^{n}(k+n)}\right)^{-1}\cdot\sum_{k=1}^{n}\frac{1}{k}\cdot\sqrt[k]{\prod_{p=1}^{k}(k+p)}\right)=$$

$$=\lim_{n\to\infty}\frac{2+\frac{\sqrt{3\cdot 4}}{2}+\frac{\sqrt[3]{4\cdot 5\cdot 6}}{3}+\cdots+\frac{\sqrt[n]{(n+1)(n+2)\dots(n+n)}}{n}}{\sqrt[n]{(n+1)(n+2)\dots(n+n)}}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{(n+1)(n+2)\dots(n+n)}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{(n+1)(n+2)\dots(n+n)}{n^n}}\overset{C.D'A}{=}$$

$$=\lim_{n\to\infty}\frac{2(n+1)(2n+1)}{(n+1)^2}\cdot\frac{n^n}{(n+1)^n}=4\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{4}{e}$$

$$\lim_{n\to\infty}\frac{2+\frac{\sqrt{3\cdot 4}}{2}+\frac{\sqrt[3]{4\cdot 5\cdot 6}}{3}+\cdots+\frac{\sqrt[n]{(n+1)(n+2)\dots(n+n)}}{n}}{n}\overset{L.C-S}{=}$$

$$=\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+2)(n+3)\dots 2n(2n+1)(2n+2)}}{n+1}=$$

$$=\lim_{n\to\infty}\sqrt[n+1]{\frac{(n+2)(n+3)\dots(n+n)(2n+1)(2n+2)}{(n+1)^{n+1}}}=$$

$$\lim_{n\to\infty}\sqrt[n]{\frac{(n+1)(n+2)\dots 2n}{n^n}}\overset{C.D'A}{=}\frac{4}{e}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\left(\left(\sqrt[n]{\prod_{k=1}^{n}(k+n)}\right)^{-1}\cdot\sum_{k=1}^{n}\frac{1}{k}\cdot\sqrt[k]{\prod_{p=1}^{k}(k+p)}\right)=1$$

1276. Find:

$$\Omega=\lim_{n\to\infty}\left(H_{2n}\cdot\int_{-1}^{1}x^{2n+1}log(1+\gamma^x)\,dx\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Remus Florin Stanca-Romania

$$I_{-} = \int_{-1}^{1} x^{2n+1}\log(1+\gamma^x)\,dx \overset{x=-t}{=} -\int_{1}^{-1} -x^{2n+1}\log\left(1+\frac{1}{\gamma^x}\right)dx =$$

$$= -\int_{-1}^{1} x^{2n+1}\log\left(\frac{\gamma^x+1}{\gamma^x}\right)dx = -\left(\int_{-1}^{1} x^{2n+1}\log(1+\gamma^x)\,dx - \int_{-1}^{1} x^{2n+2}\log\gamma\,dx\right) =$$

$$= -\int_{-1}^{1} x^{2n+1}\log(1+\gamma^x)\,dx + \int_{-1}^{1} x^{2n+2}\log\gamma\,dx \Rightarrow$$

$$2I = \int_{-1}^{1} x^{2n+2}\log\gamma\,dx = \frac{2}{2n+3}\log\gamma \Rightarrow I = \frac{1}{2n+3}\log\gamma$$

$$\Omega = \lim_{n\to\infty}\left(H_{2n}\cdot\int_{-1}^{1} x^{2n+1}\log(1+\gamma^x)\,dx\right) = \log\gamma\cdot\lim_{n\to\infty}\frac{H_{2n}-\log 2n+\log 2n}{2n+3} = 0$$

Solution 2 by Kaushik Mahanta-Assam-India

$$I_1 = \int_{-1}^{1} x^{2n+1}\log(1+\gamma^x)\,dx \overset{x=-t}{=} -\int_{1}^{-1} (-x)^{2n+1}\log(1+\gamma^{-x})\,dx =$$

$$= -\int_{-1}^{1} x^{2n+1}\log\left(\frac{\gamma^x+1}{\gamma^x}\right)dx = -\int_{-1}^{1} x^{2n+1}\log(1+\gamma^x)\,dx + \int_{-1}^{1} x^{2n+2}\log\gamma\,dx$$

$$2I_1 = 2\log\gamma\int_{0}^{1} x^{2n+2}dx = \frac{\log\gamma}{2n+3}$$

$$\Omega = \lim_{n\to\infty}\left(H_{2n}\cdot\int_{-1}^{1} x^{2n+1}\log(1+\gamma^x)\,dx\right) = \log\gamma\lim_{n\to\infty}\left(\frac{H_{2n}}{2n+3}\right) =$$

Recall, $H_{2n}\sim\gamma+\log n+o\left(\frac{1}{2n}\right)$

$$= \log\gamma\lim_{n\to\infty}\frac{\gamma+\log(2n)+o\left(\frac{1}{4n}\right)}{2n+3} = \log\gamma\lim_{n\to\infty}\frac{\log(2n)}{2n\left(1+\frac{3}{2n}\right)} = 0$$

Solution 3 by Adrian Popa-Romania

$$I=\int_{-1}^{1}x^{2n+1}log(1+\gamma^x)\,dx\overset{x=-t}{=}-\int_{1}^{-1}(-x)^{2n+1}log(1+\gamma^{-x})\,dx=$$

$$=-\int_{-1}^{1}x^{2n+1}log\left(\frac{\gamma^x+1}{\gamma^x}\right)dx=-\int_{-1}^{1}x^{2n+1}log(1+\gamma^x)\,dx+\int_{-1}^{1}x^{2n+2}log\gamma\,dx$$

$$2I=2log\gamma\int_{0}^{1}x^{2n+2}dx=\frac{log\gamma}{2n+3}$$

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\cong\gamma+logn$$

$$H_{2n}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}$$

$$\lim_{n\to\infty}H_{2n}=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}=\int_{0}^{1}\frac{1}{1+x}dx=log2$$

$$\Rightarrow H_{2n}=\gamma+log(2)$$

$$\Omega=\lim_{n\to\infty}\left(H_{2n}\cdot\int_{-1}^{1}x^{2n+1}log(1+\gamma^x)\,dx\right)=log\gamma\lim_{n\to\infty}\left(\frac{H_{2n}}{2n+3}\right)=$$

$$=log\gamma\lim_{x\to\infty}\frac{\gamma+log2x}{2x+3}=log\gamma\lim_{x\to\infty}\frac{1}{x}=0$$

1277. For $p\in\mathbb{N}^*, p\geq 2$ find:

$$\Omega=\lim_{n\to\infty}\frac{\sqrt[p]{n}-\sqrt[p]{n-1}+\sqrt[p]{n-2}-\cdots+(-1)^{n-1}\sqrt[p]{1}}{\sqrt[pn]{n!}}$$

Proposed by Marian Ursărescu-Romania

Solution by proposer

$$\Omega=\lim_{n\to\infty}\frac{\sqrt[p]{n}-\sqrt[p]{n-1}+\sqrt[p]{n-2}-\cdots+(-1)^{n-1}\sqrt[p]{1}}{\sqrt[pn]{n!}}=$$

$$=\lim_{n\to\infty}\frac{\sqrt[p]{n}-\sqrt[p]{n-1}+\sqrt[p]{n-2}-\cdots+(-1)^{n-1}\sqrt[p]{1}}{\sqrt[p]{n!}}\cdot\frac{\sqrt[p]{n!}}{\sqrt[pn]{n!}};\ (1)$$

$$\lim_{n\to\infty}\frac{\sqrt[p]{n!}}{\sqrt[pn]{n!}}=\lim_{n\to\infty}\sqrt[p]{\frac{n}{\sqrt[n]{n!}}}=\sqrt[p]{\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{n!}}}\overset{C-D'A}{=}\sqrt[p]{\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}}=$$

$$=\sqrt[p]{\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n}=\sqrt[p]{e};(2)$$

Let be $a_n=\sqrt[p]{n}-\sqrt[p]{n-1}+\sqrt[p]{n-2}-\cdots+(-1)^{n-1}\sqrt[p]{1}$ and $b_n=\sqrt[p]{n}$, hence

$$\lim_{n\to\infty}\frac{a_{2n}}{b_{2n}}\overset{LC-S}{=}\lim_{n\to\infty}\frac{a_{2n+2}-a_{2n}}{b_{2n+2}-b_{2n}}=\lim_{n\to\infty}\frac{\sqrt[p]{2n+2}-\sqrt[p]{2n+1}}{\sqrt[p]{2n+2}-\sqrt[p]{2n}}=$$

$$=\lim_{n\to\infty}\frac{\sqrt[p]{(2n+2)^{p-1}}+\cdots+\sqrt[p]{(2n)^{p-1}}}{2\left(\sqrt[p]{(2n+2)^{p-1}}+\cdots+\sqrt[p]{(2n+1)^{p-1}}\right)}=\frac{1}{2};(3)$$

$$\lim_{n\to\infty}\frac{a_{2n+1}}{b_{2n+1}}\overset{LC-S}{=}\lim_{n\to\infty}\frac{a_{2n+1}-a_{2n-1}}{b_{2n+1}-b_{2n-1}}=\lim_{n\to\infty}\frac{\sqrt[p]{2n+1}-\sqrt[p]{2n-1}}{\sqrt[p]{2n+1}-\sqrt[p]{2n-1}}=$$

$$=\lim_{n\to\infty}\frac{\sqrt[p]{(2n+1)^{p-1}}+\cdots+\sqrt[p]{(2n-1)^{p-1}}}{2\left(\sqrt[p]{(2n+1)^{p-1}}+\cdots+\sqrt[p]{(2n)^{p-1}}\right)}=\frac{1}{2};(4)$$

From (1),...,(4), we get:

$$\Omega=\lim_{n\to\infty}\frac{\sqrt[p]{n}-\sqrt[p]{n-1}+\sqrt[p]{n-2}-\cdots+(-1)^{n-1}\sqrt[p]{1}}{\sqrt[pn]{n!}}=\frac{\sqrt[p]{e}}{2}$$

1278. Prove the summation:

$$\sum_{n=1}^{\infty}\frac{(F_n+\phi^{n-1})(L_n+\phi^{n-1})}{\phi^{6n-1}}=\frac{4\phi}{15}$$

F_n –Fibonacci numbers, L_n –Lucas numbers, ϕ –Golden Ratio.

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Izumi Ainsworth-Peru

$$S=\sum_{n=1}^{\infty}\frac{(F_n+\phi^{n-1})(L_n+\phi^{n-1})}{\phi^{6n-1}}$$

Applying the following generalizations and summation formulas:

$$F_n=\frac{\phi^n-(-\phi)^{-n}}{\sqrt{5}};L_n=\phi^n+(-\phi)^{-n}$$

$$\sum_{n=1}^{\infty}F_nx^n=\frac{-x}{x^2+x-1};\sum_{n=1}^{\infty}L_nx^n=\frac{-(2x^2+x)}{x^2+x-1}$$

$$S=\frac{\phi}{\sqrt{5}}\sum_{n=1}^{\infty}\frac{\phi^{2n}-(-\phi)^{-2n}}{\phi^{6n}}+\sum_{n=1}^{\infty}F_n\left(\phi^{-5}\right)^n+\sum_{n=1}^{\infty}L_n\left(\phi^{-5}\right)^n+\frac{1}{\phi}\sum_{n=1}^{\infty}\frac{1}{\phi^{4n}}=$$

$$=\left(\frac{\phi}{\sqrt{5}}+\frac{1}{\phi}\right)\sum_{n=1}^{\infty}\frac{1}{(\phi^4)^n}-\frac{\phi}{\sqrt{5}}\sum_{n=1}^{\infty}\frac{1}{(\phi^8)^n}+\sum_{n=1}^{\infty}F_n\left(\phi^{-5}\right)^n+\sum_{n=1}^{\infty}L_n\left(\phi^{-5}\right)^n=$$

$$=\frac{\phi^2+\sqrt{5}}{\sqrt{5}\phi}\left(\frac{1}{\phi^4-1}\right)-\frac{\phi}{\sqrt{5}}\left(\frac{1}{\phi^8-1}\right)+\frac{-\phi^{-5}}{(\phi^{-5})^2+\phi^{-5}-1}+\frac{-2\left(\left(\phi^{-5}\right)^2+\phi^{-5}\right)}{(\phi^{-5})^2+\phi^{-5}-1}=$$

$$=\frac{3}{\sqrt{5}}\left(\frac{\phi^4+1}{\phi^8-1}\right)-\frac{\phi}{\sqrt{5}(\phi^8-1)}-\frac{2\phi^5+2}{1+\phi^5-\phi^{10}}=$$

$$=\frac{3\phi^4-\phi+3}{(2\phi-1)(21\phi+12)}-\frac{2(5\phi+3)+2}{1+(5\phi+3)-(55\phi+34)}=$$

$$=\frac{8\phi+9}{15(3\phi+2)}+\frac{5\phi+4}{5(5\phi+3)}=$$

$$=\frac{85\phi^2+135\phi+51}{15(15\phi^2+19\phi+6)}=\frac{220\phi+136}{15(34\phi+21)}\cdot\frac{\phi}{\phi}=\frac{(220\phi+136)\phi}{15(55\phi+34)}=\frac{4\phi}{15}$$

1279. $f(x)=\frac{2}{1+\sqrt{x}}\cdot\frac{2}{1+\sqrt{\sqrt{x}}}\cdot\frac{2}{1+\sqrt{\sqrt{\sqrt{x}}}}\dots,S=-1+\frac{1}{16}-\frac{1}{81}+\frac{1}{256}-\cdots$

Find:

$$\Omega=log\left(\lim_{x\to1}(1+f(x)(x-1))^{\frac{S}{logx}}\right)$$

Proposed by Mikael Bernardo-Mozambique

Solution 1 by Syed Shahabudeen-Kerala-India

$$f(x)=\frac{2}{1+\sqrt{x}}\cdot\frac{2}{1+\sqrt{\sqrt{x}}}\cdot\frac{2}{1+\sqrt{\sqrt{\sqrt{x}}}}\dots=\frac{logx}{x-1}(Seidels\ Formula)$$

$$S=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}=-\eta(4), where\ \eta(4)=(1-2^{-3})\zeta(4)=\frac{7\pi^4}{720}$$

Therefore,

$$\Omega=log\left(\lim_{x\to1}(1+f(x)(x-1))^{\frac{S}{logx}}\right)=log\left(\lim_{x\to1}(1+logx)^{\frac{S}{logx}}\right)=$$

$$=loge^S=-\frac{7\pi^4}{720}$$

Solution 2 by Kaushik Mahanta-Assam-India

$$\frac{logx}{x-1}=\prod_{k=1}^{\infty}\frac{2}{1+x^{\frac{1}{2^k}}}, f(x)=\frac{logx}{x-1}$$

$$S=-1+\frac{1}{2^4}-\frac{1}{3^4}+\frac{1}{4^4}-\cdots=\sum_{k=1}^{\infty}\frac{(-1)^n}{n^4}=-\sum_{k=1}^{\infty}\frac{(-1)^{n-1}}{n^4}=-(1-2^{1-4})\eta(4)=-\frac{7\pi^4}{720}$$

Therefore,

$$\Omega=log\left(\lim_{x\to1}(1+f(x)(x-1))^{\frac{S}{logx}}\right)=log\left(\lim_{x\to1}(1+logx)^{\frac{S}{logx}}\right)=loge^S=-\frac{7\pi^4}{720}$$

1280. Find:

$$\Omega=\lim_{n\to\infty}\left(n\int_0^{\frac{1}{n}}\frac{x^2}{(cosx+xsinx)^2}dx\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Igor Soposki-Skopje-Macedonia

$$I_1=\int\frac{x^2}{(cosx+xsinx)^2}dx=\int\left(-\frac{x}{cosx}\cdot\frac{-xcosx}{(xsinx+cosx)^2}\right)dx$$

$$\begin{cases}u=\frac{x}{cosx}; dv=-\frac{xcosx}{(xsinx+cosx)^2}dx\\ du=\frac{cosx-x(-sinx)}{cos^2x}=\frac{cosx+xsinx}{cos^2x}\end{cases}$$

$$v=-\int\frac{x\cos x}{(x\sin x+\cos x)^2}dx\overset{x\sin x+\cos x=t}{=}-\int\frac{dt}{t^2}=\frac{1}{t}=\frac{1}{x\sin x+\cos x}$$

$$I_1=-\frac{x}{\cos x}\frac{1}{x\sin x+\cos x}+\int\frac{1}{x\sin x+\cos x}\frac{x\sin x+\cos x}{\cos^2 x}dx=$$

$$=-\frac{x}{\cos x(x\sin x+\cos x)}+\tan x=\frac{\sin x-x\cos x}{x\sin x+\cos x}+C$$

$$\Omega=\lim_{n\to\infty}\left(nI_1\Big|_0^{\frac{1}{n}}\right)=\lim_{n\to\infty}\left(n\frac{\sin x-x\cos x}{x\sin x+\cos x}\Big|_0^{\frac{1}{n}}\right)=$$

$$=\lim_{n\to\infty}\left(n\frac{\sin\frac{1}{n}-\frac{1}{n}\cos\frac{1}{n}}{\frac{1}{n}\sin\frac{1}{n}+\cos\frac{1}{n}}\right)\overset{\frac{1}{n}=t}{=}\lim_{t\to 0}\frac{\sin t-t\cos t}{t^2\sin t+t\cos t}\overset{L'H}{=}$$

$$=\lim_{t\to 0}\frac{t\sin t}{t\sin t+t^2\cos t+\cos t}=0$$

$$\Omega=\lim_{n\to\infty}\left(n\int_0^{\frac{1}{n}}\frac{x^2}{(\cos x+x\sin x)^2}dx\right)=0$$

Solution 2 by Mohammad Rostami-Kabul-Afghanistan

$$\Omega=\lim_{n\to\infty}\left(n\int_0^{\frac{1}{n}}\frac{x^2}{(\cos x+x\sin x)^2}dx\right)=\lim_{n\to\infty}\left(\frac{\int_0^{\frac{1}{n}}\frac{x^2}{(\cos x+x\sin x)^2}dx}{\frac{1}{n}}\right)=$$

$$=\lim_{n\to\infty}\frac{-\frac{1}{n^2}\frac{\frac{1}{n^2}}{\left[\cos\frac{1}{n}+\frac{1}{n}\sin\frac{1}{n}\right]^2}}{-\frac{1}{n^2}}=\lim_{n\to\infty}\frac{1}{n^2\left[\cos\frac{1}{n}+\frac{1}{n}\sin\frac{1}{n}\right]^2}=0$$

Solution 3 by Serlea Kabay-Liberia

$$I=\int_0^{\frac{1}{n}}\frac{x^2}{(\cos x+x\sin x)^2}dx=\int_0^{\frac{1}{n}}\frac{x^2\cos x+\sec x}{(\cos x+x\sin x)^2}dx\overset{\substack{IBP\\ u=x\sec x}}{=}$$

$$=\frac{-\frac{1}{n}sec\frac{1}{n}}{\frac{1}{n}sin\frac{1}{n}+cos\frac{1}{n}}+tan\frac{1}{n}$$

$$\Omega=\lim_{n\to\infty}\left(n\int_0^{\frac{1}{n}}\frac{x^2}{(cosx+xsinx)^2}dx\right)=\lim_{n\to\infty}\left(n\frac{-\frac{1}{n}sec\frac{1}{n}}{\frac{1}{n}sin\frac{1}{n}+cos\frac{1}{n}}+ntan\frac{1}{n}\right)=0$$

Solution 4 by Mikael Bernardo-Mozambique

$$I=\int_0^{\frac{1}{n}}\frac{x^2}{(cosx+xsinx)^2}dx=\int_0^{\frac{1}{n}}\frac{\frac{x^2}{x^2cos^2x}}{\frac{(cosx+xsinx)^2}{x^2cos^2x}}dx=\int_0^{\frac{1}{n}}\frac{sec^2x}{\left(\frac{1}{x}+tanx\right)^2}dx=$$

$$=\int_0^{\frac{1}{n}}\frac{sec^2x-\frac{1}{x^2}}{\left(tanx+\frac{1}{x}\right)^2}dx+\int_0^{\frac{1}{n}}\frac{\frac{1}{x^2}}{\left(tanx+\frac{1}{x}\right)}dx=\int_0^{\frac{1}{n}}\frac{d\left(tanx+\frac{1}{x}\right)}{\left(tanx+\frac{1}{x}\right)^2}dx+\int_0^{\frac{1}{n}}\frac{dx}{(xtanx+1)^2}=$$

$$=-\frac{1}{tanx+\frac{1}{x}}\Bigg|_0^{\frac{1}{n}}+\int_0^{\frac{1}{n}}\frac{cot^2x}{(x+costx)^2}dx=-\frac{1}{tanx+\frac{1}{x}}\Bigg|_0^{\frac{1}{n}}+\frac{1}{x+cotx}\Bigg|_0^{\frac{1}{n}}=$$

$$=\left[\frac{tanx}{1+xtanx}-\frac{x}{1+xtanx}\right]_0^{\frac{1}{n}}=\frac{tan\frac{1}{n}-\frac{1}{n}}{\frac{1}{n}tan\frac{1}{n}+1}$$

Now,

$$\Omega=\lim_{n\to\infty}n\cdot I=\lim_{n\to\infty}\left(n\cdot\frac{tan\frac{1}{n}-\frac{1}{n}}{\frac{1}{n}tan\frac{1}{n}+1}\right)\overset{\frac{1}{n}=y}{=}\lim_{y\to 0}\left(\frac{1}{y}\frac{tany-y}{1+ytany}\right)=$$

$$=\lim_{y\to 0}\frac{\frac{tany}{y}-1}{1+ytany}=0$$

Solution 5 by Ravi Prakash-New Delhi-India

Let $f(x)=cosx+xsinx, f'(x)=xcosx>0,\forall 0<x<1\Rightarrow f$ –strictly increasing on $[0,1]$ and in particular on $\left[0,\frac{1}{n}\right],\forall n\geq 1\Rightarrow f(0)\leq f(x)\leq f\left(\frac{1}{n}\right),\forall x\in\left[0,\frac{1}{n}\right]\Rightarrow$

$$1 \le f(x) \le \cos\frac{1}{n} + \frac{1}{n}\sin\frac{1}{n} < 1 + \frac{1}{n^2}, \forall x \in \left[0, \frac{1}{n}\right] \Rightarrow$$

$$\frac{x^2}{\left(1+\frac{1}{n^2}\right)^2} \le \frac{x^2}{f^2(x)} \le x^2, \forall x \in \left[0, \frac{1}{n}\right] \Rightarrow$$

$$n\int_0^{\frac{1}{n}} \frac{x^2}{\left(1+\frac{1}{n^2}\right)^2} dx \le n\int_0^{\frac{1}{n}} \frac{x^2}{f^2(x)} dx \le n\int_0^{\frac{1}{n}} x^2 dx, \forall n \ge 1 \Rightarrow$$

$$\frac{1}{n^2}\frac{1}{\left(1+\frac{1}{n^2}\right)^2} \le n\int_0^1 \frac{x^2}{f^2(x)} dx \le \frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{1}{n^2}\frac{1}{\left(1+\frac{1}{n^2}\right)^2} = \lim_{n\to\infty}\frac{1}{n^2} = 0,$$

Therefore,

$$\Omega = \lim_{n\to\infty}\left(n\int_0^{\frac{1}{n}} \frac{x^2}{(\cos x + x\sin x)^2} dx\right) = 0$$

1281.

$$\Omega(a,n) = \int_{2a}^{3a} \frac{\cos^{n-1}\left(\frac{x+a}{2}\right)}{\sin^{n+1}\left(\frac{x-a}{2}\right)} dx, a > 0, n \in \mathbb{N}, n \ge 1. \text{Find:}$$

$$\Omega = \lim_{n\to\infty}\left(n \cdot \Omega\left(\frac{\pi}{3}, n\right)\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Ravi Prakash-New Delhi-India

$$\Omega(a,n) = \int_{2a}^{3a} \frac{\cos^{n-1}\left(\frac{x+a}{2}\right)}{\sin^{n+1}\left(\frac{x-a}{2}\right)} dx \overset{\frac{x-a}{2}=\theta}{=} 2\int_{\frac{a}{2}}^{a} \frac{\cos^{n-1}(\theta+a)}{\sin^{n+1}\theta} d\theta =$$

$$= 2\int_{\frac{a}{2}}^{a}\frac{(cos\theta cosa - sin\theta sina)^{n-1}}{sin^{n+1}\theta}\frac{d\theta}{sin^2\theta} =$$

$$= 2\int_{\frac{a}{2}}^{a}\frac{(cosacos\theta - sina)^{n-1}}{1}csc^2\theta\, d\theta = \left[-\frac{2}{ncosa}(cosacos\theta - sina)^n\right]_{\frac{a}{2}}^{a} =$$

$$= \frac{2}{ncosa}\left[\left(\frac{cos\frac{\pi}{2}}{sin\frac{\pi}{6}}\right)^n - \left(\frac{cos\frac{2\pi}{3}}{sin\frac{\pi}{3}}\right)^n\right] = -4\left(-\frac{1}{\sqrt{3}}\right)^n$$

Therefore,

$$\Omega = \lim_{n\to\infty}\left(n\cdot\Omega\left(\frac{\pi}{3},n\right)\right) = -4\lim_{n\to\infty}\left(-\frac{1}{\sqrt{3}}\right)^n = 0$$

1282. Find:

$$\Omega = \left(\sum_{n=0}^{\infty}\int_0^{\frac{\pi}{2}} sin^n x\cdot sinnx\, dx\right)\left(\sum_{n=0}^{\infty}\int_0^{\pi} cos^n x\cdot cosnx\, dx\right)^{-1}$$

Proposed by Daniel Sitaru-Romania

Solution by Ravi Prakash-New Delhi-India

$$I_n = \int_0^{\pi} sin^n x\cdot sinnx\, dx = \int_0^{\pi}\left(sin(\pi - x)\right)^n sin(n\pi - nx)\, dx$$

$$= \int_0^{\pi} sin^n x(-1)^{n+1}sin(nx)dx = (-1)^{n+1}I_n$$

$I_n = 0$ if n is even. Now,

$$I_{2n+1} = \int_0^{\pi} sin^{2n+1}x\cdot sin(2n+1)x\, dx = -\frac{1}{2n+1}sin^{2n+1}xcos(2n+1)x\Big|_0^{\pi} +$$

$$+\frac{2n+1}{2n+1}\int_0^{\pi} sin^{2n}xcosxcos(2n+1)x\, dx = \int_0^{\pi} sin^{2n}xcosxcos(2n+1)x\, dx =$$

$$=\int_0^{\pi} sin^{2n}x[cos(2nx)-sin(2n+1)xsinx]dx = J_n - I_{2n+1} \Rightarrow 2I_{n+1} = J_n$$

$$J_n = \int_0^{\pi} sin^{2n}xcos(2nx)dx = \frac{1}{2n}sin(2nx)sin^{2n}x\Big|_0^{\pi} - \int_0^{\pi} sin(2nx)sin^{2n-1}xcosx\,dx =$$

$$=-\int_0^{\pi} sin^{2n-1}x[sin(2n-1)x + cos(2nx)sinx]dx = -I_{2n-1} - J_n \Rightarrow J_n = -\frac{1}{2}I_{2n-1}$$

Thus, $I_{2n+1} = -\frac{1}{4}I_{2n-1}$. Also, $I_{2n} = 0, \forall n \geq 0 \Rightarrow I_{2n+1} = \left(-\frac{1}{4}\right)^n I_1$,

$$I_1 = \int_0^{\pi} sin^2x\,dx = 2\int_0^{\frac{\pi}{2}} sin^2x\,dx = \frac{\pi}{2}.$$

Thus,

$$\sum_{n=0}^{\infty} I_n = \sum_{n=0}^{\infty} I_{2n+1} = \sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n I_1 = \frac{2\pi}{5}$$

$$Let: K_n = \int_0^{\pi} cos^n x \cdot cosnx\,dx = \frac{1}{n}cos^n xsin(nx)|_0^{\pi} + \frac{n}{n}\int_0^{\pi} cos^{n-1}xsin(nx)sinx\,dx =$$

$$=\int_0^{\pi} cos^{n-1}[cos(n-1)x - cosxcos(nx)]\,dx = K_{n-1} - K_n \Rightarrow K_n = \frac{1}{2}K_{n-1}, \forall n \geq 1$$

$$K_0 = \int_0^{\pi} dx = \pi, K_n = \left(\frac{1}{2}\right)^n K_0 = \left(\frac{1}{2}\right)^n \pi \Rightarrow \sum_{n=0}^{\infty} K_n = \pi$$

Therefore,

$$\Omega = \left(\sum_{n=0}^{\infty}\int_0^{\frac{\pi}{2}} sin^n x \cdot sinnx\,dx\right)\left(\sum_{n=0}^{\infty}\int_0^{\pi} cos^n x \cdot cosnx\,dx\right)^{-1} = \frac{1}{5}$$

1283. Find:

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\frac{\omega_k}{2\omega_{k+1}}-\log n\right),\omega_n=\lim_{x\to\frac{\pi}{2}}\prod_{k=1}^{n}\frac{1-\sin^k x}{\cos^2 x}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Remus Florin Stanca-Romania

$$\lim_{x\to\frac{\pi}{2}}\prod_{k=1}^{n}\frac{1-\sin^k x}{\cos^2 x}=\lim_{x\to\frac{\pi}{2}}\frac{(1-\sin x)^n\cdot n!}{\cos^{2n}x}=n!\cdot\lim_{x\to\frac{\pi}{2}}\left(\frac{1-\sin x}{\cos^2 x}\right)^n=$$

$$=n!\cdot\lim_{x\to\frac{\pi}{2}}\left(\frac{1-\sin x}{(1-\sin x)(1+\sin x)}\right)^n=\frac{n!}{2^n}=\omega_n$$

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\frac{\omega_k}{2\omega_{k+1}}-\log n\right)=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\frac{k!}{2^k}}{\frac{(k+1)!}{2^{k+1}}2}-\log n\right)=$$

$$=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{k!}{2^k}\cdot\frac{2^k}{(k+1)!}-\log n\right)=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k+1}-\log n\right)=\gamma-1$$

Solution 2 by George Florin Şerban-Romania

$$\omega_n=\lim_{x\to\frac{\pi}{2}}\prod_{k=1}^{n}\frac{1-\sin^k x}{\cos^2 x}=\lim_{x\to\frac{\pi}{2}}\prod_{k=1}^{n}\frac{(1-\sin x)(1+\sin x+\cdots+\sin^{k-1}x)}{\cos^2 x}=$$

$$=\lim_{x\to\frac{\pi}{2}}\prod_{k=1}^{n}\frac{1-\sin x}{\cos^2 x}\cdot k\overset{L'H}{=}\lim_{x\to\frac{\pi}{2}}\prod_{k=1}^{n}k\frac{\cos x}{2\sin x\cos x}=\frac{n!}{2^n}$$

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\frac{\omega_k}{2\omega_{k+1}}-\log n\right)=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\frac{k!}{2^k}}{\frac{(k+1)!}{2^{k+1}}2}-\log n\right)=$$

$$=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{k!}{2^k}\cdot\frac{2^k}{(k+1)!}-\log n\right)=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k+1}-\log n\right)=\gamma-1$$

Solution 3 by Kaushik Mahanta-Assam-India

$$\omega_n = \lim_{x\to\frac{\pi}{2}} \prod_{k=1}^{n} \frac{1-\sin^k x}{\cos^2 x} = \lim_{x\to\frac{\pi}{2}} \left(\frac{1-\sin x}{\cos^2 x}\right)\left(\frac{1-\sin^2 x}{\cos^2 x}\right)\cdots\left(\frac{1-\sin^n x}{\cos^2 x}\right) \overset{L'H}{=}$$

$$= \lim_{x\to\frac{\pi}{2}} \left(\frac{1}{2\sin x}\right)\left(\frac{3}{2\sin x}\right)\cdots\left(\frac{n}{2\sin x}\right) = \frac{n!}{2^n}$$

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\frac{\omega_k}{2\omega_{k+1}} - \log n\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\frac{k!}{2^k}}{\frac{(k+1)!}{2^{k+1}}2} - \log n\right) =$$

$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{k!}{2^k}\cdot\frac{2^k}{(k+1)!} - \log n\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k+1} - \log n\right) = \gamma - 1$$

1284. Prove that:

$$\Omega = \lim_{n\to\infty}\left(\lim_{x\to 0}\left(\frac{(\tan x)^{(\tan x)^{(\tan x)^{\cdots}}}}{\underbrace{x^{x^{x^{\cdots}}}}_{n-times}}\right)\right) = 1$$

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Mikael Bernardo-Mozambique

Taking the Lambert function limit representation

$$W(-\log x) = -\log x \lim_{n\to\infty} \underbrace{x^{x^{x^{\cdots}}}}_{n-times}$$

$$W(-\log(\tan x)) = -\log(\tan x)\lim_{n\to\infty}(\tan x)^{(\tan x)^{(\tan x)^{\cdots}}} \Rightarrow$$

$$\Omega = \lim_{x\to 0}\left(\frac{-\log(\tan x)\lim_{n\to\infty}(\tan x)^{(\tan x)^{(\tan x)^{\cdots}}}}{-\log x \lim_{n\to\infty} \underbrace{x^{x^{x^{\cdots}}}}_{n-times}}\right) =$$

$$= \lim_{x\to 0}\left(\frac{\log x}{\log(\tan x)}\cdot\frac{W(-\log(\tan x))}{W(-\log x)}\right)$$

$$= \lim_{x\to 0}\left(\frac{\log x}{\log(\tan x)}\right)\cdot\lim_{x\to 0}\left(\frac{W(-\log(\tan x))}{W(-\log x)}\right) = \Omega_1\cdot\Omega_2$$

$$\Omega_1=\lim_{x\to 0}\frac{\frac{1}{x}}{\frac{sec^2x}{tanx}}=\lim_{x\to 0}\frac{tanx}{x}\cdot\frac{1}{sec^2x}=1$$

$$\Omega_2=\lim_{x\to 0}\left(\frac{W\left(-log(tanx)\right)}{W(-logx)}\right)\overset{L'H}{=}\lim_{x\to 0}\left(\frac{\frac{1}{e^{W\left(-log(tanx)\right)}-log(tanx)}}{\frac{1}{e^{W(-lgx)}-log(x)}}\right)=$$

$$=\lim_{x\to 0}\left(\frac{e^{W(-logx)}-logx}{e^{W\left(-log(tanx)\right)}-log(tanx)}\right)\overset{e^{W(z)}=\frac{z}{W(z)}}{=}$$

$$=\lim_{x\to}\left(\frac{-\frac{logx}{W(-logx)}-logx}{-\frac{log(tanx)}{W\left(-log(tanx)\right)}-log(tanx)}\right)=\lim_{x\to 0}\frac{logx}{log(tanx)}\cdot\frac{\frac{1}{W(-logx)}+1}{\frac{1}{W\left(-log(tanx)\right)}+1}=1$$

Therefore,

$$\Omega=\lim_{n\to\infty}\left(\lim_{x\to 0}\left(\frac{(tanx)^{(tanx)^{(tanx)^{\cdots}}}}{\underbrace{x^{x^{\cdots}}}_{n-times}}\right)\right)=1$$

Solution 2 by Hasan Mamadov-Azerbaijan

$$logI_{k+1}=\lim_{x\to 0}\left(\frac{tanxlogf_k(x)}{xlogg_k(x)}\right)=\lim_{x\to 0}[log(f_k(x)^{tanx})-log(g_k(x)^x)]$$

For $k=1$ by above proof was true. Suppose it is true for all $n=1,2,\dots,k$, which means

$$I_k=\lim_{x\to 0}\left(log\frac{f_k(x)}{g_k(x)}\right)=0,\forall n=1,2\dots,k\Rightarrow$$

$$logI_{k+1}=\lim_{x\to 0}\left[log\left(\frac{f_k(x)^{tanx}}{g_k(x)^x}\right)\right]=\lim_{x\to 0}[tanxlog\left(f_k(x)\right)-xlog\left(g_k(x)\right)]=$$

$$=\lim_{x\to 0}\{tanx[logf_k(x)-logg_k(x)]+tanxlogg_k(x)\}=0$$

Since, $logf_k(x)-logg_k(x)=0$ by induction hypothesis and $log\left(xg_k(x)\right)\overset{x\to 0}{\longrightarrow}0$.

Indeed, this can is prove by induction follows as:

$$\lim_{x\to 0}log\left(xlogg_1(x)\right)=\lim_{x\to 0}(xlogx)=\lim_{x\to 0}\frac{logx}{\frac{1}{x}}=\lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=0$$

Suppose that it is true for all $n = 1, 2, \dots, k$, then

$$\lim_{x\to 0}\left(x\log g_{k+1}(x)\right) = \lim_{x\to 0}\left(\log g_k(x)\right) = 0$$

Thus, we prove that $\log I_{k+1} = 0$ is true, so by induction principle $\log I_n = 0, \forall n \ge 1$.

Therefore,

$$\Omega = \lim_{n\to\infty}\left(\lim_{x\to 0}\left(\frac{(\tan x)^{(\tan x)^{(\tan x)^{\cdots}}}}{\underbrace{x^{x^{x^{\cdots}}}}_{n-times}}\right)\right) = 1$$

1285. If $0 < a \le b$ then:

$$\left(\int_a^b e^{x^2}dx\right)^4 \le \left(\int_a^b x^3 e^{x^2}dx\right)\left(\int_a^b \frac{e^{x^2}}{x}dx\right)^3$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$\int_a^b f(x)g(x)dx \le \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b |g(x)|^q\right)^{\frac{1}{q}}; p, q > 0, \frac{1}{p} + \frac{1}{q} = 1\ (Holder)$$

For $p = 4, q = \frac{4}{3} \to \frac{1}{p} + \frac{1}{q} = \frac{1}{4} + \frac{3}{4} = 1, f(x) = \sqrt[4]{x^3 e^{x^2}}; g(x) = \sqrt[4]{\left(\frac{e^{x^2}}{x}\right)^3} \to$

$$\left(\int_a^b \left(\sqrt[4]{x^3 e^{x^2}}\right)^4\right)^{\frac{1}{4}} \cdot \left(\int_a^b \left(\sqrt[4]{\left(\frac{e^{x^2}}{x}\right)^3}\right)^{\frac{4}{3}}\right)^{\frac{3}{4}} \ge \int_a^b \sqrt[4]{x^3 e^{x^2}} \cdot \sqrt[4]{\left(\frac{e^{x^2}}{x}\right)^3}\, dx \to$$

$$\left(\int_a^b x^3 e^{x^2}dx\right)^{\frac{1}{4}}\left(\int_a^b \frac{e^{x^2}}{x}dx\right)^{\frac{3}{4}} \ge \int_a^b e^{x^2}\, dx \Leftrightarrow$$

$$\left(\int_a^b e^{x^2}dx\right)^4 \le \left(\int_a^b x^3 e^{x^2}dx\right)\left(\int_a^b \frac{e^{x^2}}{x}dx\right)^3$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

For $0 < a \leq b$. Let us denote: $f(x) = \frac{e^{x^2}}{x}; g(x) = x^3 e^{x^2}, \forall x \in [a,b]$

$$\left(\int_a^b g(x)dx\right)\left(\int_a^b f(x)dx\right)^3 = \left(\int_a^b g(x)dx\right)\left(\int_a^b f(x)dx\right)\left(\int_a^b f(x)dx\right)\left(\int_a^b f(x)dx\right) \overset{BCS}{\geq}$$

$$\geq \left(\int_a^b g(x)f(x)dx\right)^2\left(\int_a^b (f(x))^2 dx\right)^2 = \left(\int_a^b g(x)f(x)dx \cdot \int_a^b (f(x))^2 dx\right)^2 \overset{BCS}{\geq}$$

$$\overset{BCS}{\geq} \left(\int_a^b g(x)(f(x))^3 dx\right)^4 = \left(\int_a^b e^{4x^2} dx\right)^4 \overset{e^{4x^2} \geq e^{x^2}}{\geq} \left(\int_a^b e^{x^2} dx\right)^4$$

1286. **Prove without any software:**

$$\int_0^1 \frac{x^2}{e^{2x^2}} dx + \frac{1}{2}\int_0^1 \frac{1}{e^{2x^2}} dx > \frac{1}{4e^2}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\int_0^1 \frac{x^2}{e^{2x^2}} dx + \frac{1}{2}\int_0^1 \frac{1}{e^{2x^2}} dx \overset{AM-GM}{\geq} \int_0^1 \frac{2\sqrt{2}x}{2e^{2x^2}} dx = \frac{\sqrt{2}}{4}\left[-\frac{1}{e^{2x^2}}\right]_0^1 = \frac{\sqrt{2}}{4}\left(1 - \frac{1}{e^2}\right)$$

Now,

$\frac{\sqrt{2}}{4}\left(1 - \frac{1}{e^2}\right) \geq \frac{1}{4e^2} \Leftrightarrow e^2 \geq 1 + \frac{\sqrt{2}}{2}$, which is true because $e^2 > 2 > 1 + \frac{\sqrt{2}}{2}$

Solution 2 by Gabriel Brehuescu-Romania

$$\frac{1}{4e^2} = \left(\frac{x}{4e^{2x^2}}\right)_0^1 = \int_0^1 \left(\frac{x}{4e^{2x^2}}\right)' dx = \int_0^1 \frac{1 - 4x^2}{4e^{2x^2}} dx$$

$$\int_0^1 \frac{x^2}{e^{2x^2}} dx + \frac{1}{2}\int_0^1 \frac{1}{e^{2x^2}} dx > \frac{1}{4e^2}$$

$$\Leftrightarrow \int_0^1 \left(\frac{x^2}{e^{2x^2}} + \frac{1}{2e^{2x^2}}\right) dx > \int_0^1 \frac{1 - 4x^2}{4e^{2x^2}} dx; (*)$$

$$\because \frac{2x^2 + 1}{2e^{2x^2}} > \frac{1 - 4x^2}{4e^{2x^2}}, \forall x \in [0,1] \to (*)\ true.$$

Solution 3 by Adrian Popa-Romania

$$\int_0^1 \frac{2x^2+1}{2e^{2x^2}}dx > \int_0^1 \frac{1}{4e^2}dx \Leftrightarrow \frac{2x^2+1}{2e^{2x^2}} > \frac{1}{4e^2} \Leftrightarrow \frac{2x^2+1}{e^{2x^2}} > \frac{1}{2e^2}, \forall x \in [0,1]$$

Let $2x^2 = t \to t \in [0,2]$. We must to prove: $\frac{t+1}{e^t} > \frac{1}{2e^2}, \forall t \in [0,2]$

$$f(t) := \frac{t+1}{e^t} \to f'(t) = -\frac{t}{e^t}; f'(t) = 0 \leftrightarrow t = 0$$

$$\to f \text{ is decreasing on } [0,2] \to f(1) > \frac{3}{e^2}$$

Solution 4 by Satyam Roy-India

$$\int_0^1 \frac{x^2}{e^{2x^2}}dx + \frac{1}{2}\int_0^1 \frac{1}{e^{2x^2}}dx = \int_0^1 \frac{2x^2+1}{2e^{2x^2}}dx \overset{AM-GM}{\geq} \int_0^1 \frac{2\sqrt{2x^2\cdot 1}}{2e^{2x^2}}dx = \int_0^1 \frac{\sqrt{2}x}{e^{2x^2}}dx \overset{t=x^2}{=}$$

$$= \frac{\sqrt{2}}{2}\int_0^1 \frac{dt}{e^{2t}} = \frac{\sqrt{2}}{4}\left(1 - \frac{1}{e^2}\right)$$

Now, $\frac{\sqrt{2}}{4}\left(1-\frac{1}{e^2}\right) \geq \frac{1}{4e^2}$ true from $e^2 \geq 1 + \frac{\sqrt{2}}{2}$.

1287. If $a, b \geq 0$ then:

$$(a+b)\sqrt{e^{-2a\int_0^\infty \frac{\log x}{e^x}dx} + e^{-2b\int_0^1 \log\left(\log\frac{1}{x}\right)dx}} \geq \sqrt{2}(2ab\gamma + a + b)$$

Proposed by Daniel Sitaru-Romania

Solution by Mohammad Rostami-Kabul-Afghanistan

$$I_1 = \int_0^\infty \frac{\log x}{e^x}dx = \int_0^\infty e^{-x}\log(x)dx \overset{e^{-x}=u}{=} \int_0^1 \log\left(\log\left(\frac{1}{u}\right)\right)du = I_2$$

$$I_1 = \int_0^\infty \frac{\log x}{e^x}dx = \int_0^\infty e^{-x}\frac{\partial}{\partial a}\bigg|_{a=0} x^a dx = \frac{\partial}{\partial a}\bigg|_{a=0}\int_0^\infty e^{-x}x^{(a+1)-1}dx =$$

$$= \frac{\partial}{\partial a}\bigg|_{a=0}\Gamma(a+1) = [\Gamma(a+1)\cdot\psi(a+1)]_{a=0} = \Gamma(1)\psi(1) = -\gamma$$

Hence, $I_1 = I_2 = -\gamma$

$$\begin{cases} f(x) = e^x \\ g(x) = x+1 \end{cases} \Rightarrow \begin{cases} f'(x) = e^x \\ g'(x) = 1 \end{cases} \overset{x\geq 0}{\Longrightarrow} e^x \geq 1 \Rightarrow f(x) \geq g(x) \Rightarrow e^x \geq x+1, \forall x \in \mathbb{R}; (1)$$

By AQ-AM inequality $\sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2}$, we have:

$$\sqrt{\frac{e^{-2aI_1}+e^{-2bI_2}}{2}} = \sqrt{\frac{(e^{a\gamma})^2+(e^{b\gamma})^2}{2}} \geq \frac{e^{a\gamma}+e^{b\gamma}}{2} \overset{(1)}{\Rightarrow}$$

$$\sqrt{\frac{e^{-2aI_1}+e^{-2bI_2}}{2}} \geq \frac{(a\gamma+1)+(b\gamma+1)}{2} = \frac{(a+b)\gamma+2}{2} =$$

$$= \left(\frac{a+b}{2}\right)\gamma+1 \geq \frac{2ab}{a+b}\gamma+1 \Rightarrow$$

$$(a+b)\sqrt{e^{2a\gamma}+e^{2b\gamma}} \geq \sqrt{2}(2ab\gamma+a+b)$$

1288. Find:

$$\Omega = \int \frac{sin^{-1}\left(\sqrt{\frac{2}{x+3}}\right)+tan^{-1}\left(\sqrt{\frac{2}{x+1}}\right)}{x\left(\frac{\pi}{2}-sin^{-1}\left(\sqrt{\frac{x-1}{x+3}}\right)\right)}dx$$

Proposed by Daniel Sitaru-Romania

Solution by Ravi Prakash-New Delhi-India

Let $\alpha = sin^{-1}\left(\sqrt{\frac{2}{x+3}}\right); \beta = tan^{-1}\left(\sqrt{\frac{2}{x+1}}\right); 0 < \frac{2}{x+3} \leq 1, x+3 > 0 \Rightarrow$

$$x+3 \geq 2 \Rightarrow x \geq -1$$

$$tan^2\beta = \frac{2}{x+1} \Rightarrow 1+tan^2\beta = \frac{x+3}{x+1} \Rightarrow sec^2\beta = \frac{x+3}{x+1} \Rightarrow cos\beta = \sqrt{\frac{x+1}{x+3}}$$

Also, $sin\beta = tan\beta cos\beta = \sqrt{\frac{2}{x+1}}\sqrt{\frac{x+1}{x+3}} = \sqrt{\frac{2}{x+3}} = sin\alpha \Rightarrow \alpha = \beta$.

Next, $\frac{\pi}{2} - sin^{-1}\left(\sqrt{\frac{x-1}{x+3}}\right) = cos^{-1}\left(\sqrt{\frac{x-1}{x+3}}\right)$

$$sin^2\alpha = \frac{2}{x+3} \Rightarrow cos^2\alpha = \frac{x-1}{x+3} \Rightarrow cos\alpha = \sqrt{\frac{x-1}{x+3}} \Rightarrow \alpha = cos^{-1}\left(\sqrt{\frac{x-1}{x+3}}\right)$$

Therefore,

$$\Omega=\int\frac{\sin^{-1}\left(\sqrt{\frac{2}{x+3}}\right)+\tan^{-1}\left(\sqrt{\frac{2}{x+1}}\right)}{x\left(\frac{\pi}{2}-\sin^{-1}\left(\sqrt{\frac{x-1}{x+3}}\right)\right)}dx=$$

$$=\int\frac{\sin^{-1}\left(\sqrt{\frac{2}{x+3}}\right)+\tan^{-1}\left(\sqrt{\frac{2}{x+1}}\right)}{x\cos^{-1}\left(\sqrt{\frac{x-1}{x+3}}\right)}dx=\int\frac{2}{x}dx=2\log|x|+C$$

1289. Find without any software:

$$\Omega=\int\frac{1+(1-x^2-e^x)e^x}{(1+xe^x)\sqrt{(1-x^2)(1-e^{2x})}}dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$\Omega=\int\frac{1+(1-x^2-e^x)e^x}{(1+xe^x)\sqrt{(1-x^2)(1-e^{2x})}}dx=\int\frac{1+e^x-x^2e^x-e^{2x}}{(1+xe^x)\sqrt{1-e^{2x}-x^2+x^2e^{2x}}}dx=$$

$$=\int\frac{1+e^x-x^2e^x-e^{2x}}{(1+xe^x)\sqrt{(1+xe^x)-(x+e^x)^2}}dx=\int\frac{1+e^x-x^2e^x-e^{2x}}{(1+xe^x)^2\sqrt{1-\left(\frac{x+e^x}{1+xe^x}\right)^2}}dx=$$

$$\overset{\frac{x+e^x}{1+xe^x}=t}{=}\int\frac{dt}{\sqrt{1-t^2}}=\sin^{-1}t+C=\sin^{-1}\left(\frac{x+e^x}{1+xe^x}\right)+C$$

Solution 2 by Yen Tung Chung-Taichung-Taiwan

$$\Omega=\int\frac{1+(1-x^2-e^x)e^x}{(1+xe^x)\sqrt{(1-x^2)(1-e^{2x})}}dx=\int\frac{1+e^x-x^2e^x-e^{2x}}{(1+xe^x)\sqrt{1-e^{2x}-x^2+x^2e^{2x}}}dx=$$

$$=\int\frac{1+e^x-x^2e^x-e^{2x}}{(1+xe^x)\sqrt{(1+xe^x)-(x+e^x)^2}}dx=$$

$$=\int\frac{1}{\sqrt{1-\left(\frac{x+e^x}{1+xe^x}\right)^2}}\cdot\frac{1+e^x-x^2e^x-e^{2x}}{(1+xe^x)^2}dx=$$

$$=\int\frac{1}{\sqrt{1-\left(\frac{x+e^x}{1+xe^x}\right)^2}}\cdot d\left(\frac{x+e^x}{1+xe^x}\right)=\sin^{-1}\left(\frac{x+e^x}{1+xe^x}\right)+C$$

Solution 3 by Hafiz Iqbal-Situbondo-Indonesia

$$\Omega=\int\frac{1+(1-x^2-e^x)e^x}{(1+xe^x)\sqrt{(1-x^2)(1-e^{2x})}}dx=\int\frac{1+e^x-x^2e^x-e^{2x}}{(1+xe^x)\sqrt{1-e^{2x}-x^2+x^2e^{2x}}}dx=$$

$$=\int\frac{1+e^x-x^2e^x-e^{2x}}{(1+xe^x)\sqrt{(1+xe^x)-(x+e^x)^2}}dx=\int\frac{1+e^x-x^2e^x-e^{2x}}{(1+xe^x)^2\sqrt{1-\left(\frac{x+e^x}{1+xe^x}\right)^2}}dx=$$

$$\overset{\frac{x+e^x}{1+xe^x}=u}{=}\int\frac{du}{\sqrt{1-u^2}}=\sin^{-1}u+C=\sin^{-1}\left(\frac{x+e^x}{1+xe^x}\right)+C$$

1290. Find:

$$\Omega=\int_0^x\frac{t^2}{(t\cdot\sinh t-\cosh t)^2}dt$$

Proposed by Daniel Sitaru-Romania

Solution by Mikael Bernardo-Mozambique

$$\Omega=\int_0^x\frac{t^2}{(t\cdot\sinh t-\cosh t)^2}dt$$

$$I=\int_0^x\frac{t^2}{(t\cdot\sinh t-\cosh t)^2}dt=\int_0^x\frac{\frac{t^2}{t^2\cosh^2t}}{\frac{(t\sinh t-\cosh t)^2}{t^2\cosh^2 t}}dt=$$

$$=\int_0^x\frac{\operatorname{sech}^2t+\frac{1}{t^2}-\frac{1}{t^2}}{\left(\tanh t-\frac{1}{t}\right)^2}dt=\int_0^x\frac{\operatorname{sech}^2t+\frac{1}{t^2}}{\left(\tanh t-\frac{1}{t}\right)^2}dt-\int_0^x\frac{\frac{1}{t^2}}{\left(\tanh t-\frac{1}{t}\right)^2}dt=$$

$$=\int_0^x\frac{d\left(\tanh t-\frac{1}{t}\right)}{\left(\tanh t-\frac{1}{t}\right)^2}dt-\int_0^x\frac{dt}{t^2\left(\tanh t-\frac{1}{t}\right)^2}=-\frac{1}{\tanh t-\frac{1}{t}}\Bigg|_0^x-\int_0^x\frac{dt}{(t(\tanh t)-1)^2}$$

$$I=-\frac{1}{tanhx-\frac{1}{x}}-\int_0^x\frac{coth^2t}{(t-coth)^2}dt$$

$$I_1=\int_0^x\frac{coth^2t}{(t-coth)^2}dt=\int_0^x\frac{cosh^2t}{(cotht-1)^2}dt\overset{IBP}{=}$$

$$=\frac{t-cotht}{(cotht-t)^2}\Big|_0^x-2\int_0^x\frac{(cotht-t)(-csch^2t-1)dt}{(cotht-t)^3}=$$

$$=\frac{1}{t-cotht}\Big|_0^x-2\int_0^x\frac{d(cotht-t)}{(cotht-t)^2}=\frac{1}{t-cotht}\Big|_0^x-\frac{2}{t-cotht}\Big|_0^x=\frac{1}{t-cotht}\Big|_0^x=$$

$$=\frac{tanht}{1-t\cdot tanht}\Big|_0^x=\frac{tanhx}{1-xtanhx}$$

1291. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{e^n}\int_0^n\frac{dx}{e^x\left(e^{x+2}+2e-\frac{1}{e}\right)}\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mikael Bernardo-Mozambique

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{e^n}\int_0^n\frac{dx}{e^x\left(e^{x+2}+2e-\frac{1}{e}\right)}\right)=\lim_{n\to\infty}\left(\frac{1}{e^n}\int_0^n\frac{dx}{e^xe^x\left(1+e^{-x}\left(\frac{2}{e}-\frac{1}{e^3}\right)\right)}\right)=$$

$$=\lim_{n\to\infty}\left(\frac{1}{e^2e^n}\int_0^n\frac{e^{-x}e^{-x}dx}{\left(1+e^{-x}\left(\frac{2}{e}-\frac{1}{e^3}\right)\right)}\right)\overset{e^{-x}=y}{=}\lim_{n\to\infty}\left(\frac{1}{e^2e^n}\int_{\frac{1}{e^n}}^1\frac{ydy}{1+y\left(\frac{2}{e}-\frac{1}{e^3}\right)}\right)\overset{a=\frac{2}{e}-\frac{1}{e^3}}{=}$$

$$=\lim_{n\to\infty}\left(\frac{1}{e^2e^n}\int_{\frac{1}{e^n}}^1\frac{ydy}{ay+1}\right)=\lim_{n\to\infty}\left(\frac{1}{e^2e^n}\cdot\frac{1}{a}\int_{\frac{1}{e^n}}^1\left(1-\frac{1}{ay+1}\right)dy\right)=$$

$$= \lim_{n\to\infty}\left(\frac{1}{e^2e^n}\left[y-\frac{1}{a}log(ay+1)\right]_{\frac{1}{e^n}}^{1}\right)=$$

$$= \lim_{n\to\infty}\left(\frac{1}{e^2e^n}\cdot\frac{1}{a}\left\{(1-\frac{1}{a}log(a+1)-\left(\frac{1}{e^n}-\frac{1}{a}log\left(\frac{a}{e^n}+1\right)\right)\right\}\right)=$$

$$= 0\cdot\frac{1}{e^2}\cdot\frac{1}{a}\left\{1-\frac{1}{a}log(a+1)-0+\frac{1}{a}log(0+1)\right\}=0.$$

Solution 2 by Mohammad Rostami-Kabul-Afghanistan

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{e^n}\int_0^n\frac{dx}{e^x\left(e^{x+2}+2e-\frac{1}{e}\right)}\right);I=\int_0^n\frac{dx}{e^x\left(e^{x+2}+2e-\frac{1}{e}\right)}\overset{e^x=t}{=}$$

$$=\int_1^{e^n}\frac{1}{t\left(e^2t+2e-\frac{1}{e}\right)}\frac{dt}{t}=\int_1^{e^n}\frac{dt}{t^2\left(e^2t+2e-\frac{1}{e}\right)}=$$

$$=-\frac{e^4}{(2e^2-1)^2}\int_1^{e^n}\frac{1}{t}dt+\frac{e}{2e^2-1}\int_1^{e^n}\frac{1}{t^2}dt+\frac{e^4}{(2e^2-1)^2}\int_1^{e^n}\frac{e^2}{e^2t+2e-\frac{1}{e}}dt=$$

$$=-\frac{e^4}{(2e^2-1)^2}[logt]_1^{e^n}+\frac{e}{2e^2-1}\left[-\frac{1}{t}\right]_1^{e^n}+\frac{e^4}{(2e^2-1)^2}\left[log\left(e^2t+2e-\frac{1}{e}\right)\right]_1^{e^n}=$$

$$=-\frac{e^4}{(2e^2-1)^2}\cdot n+\frac{e}{(2e^2-1)}\left(1-\frac{1}{e^n}\right)+\frac{e^4}{(2e^2-1)^2}log\left(\frac{e^2e^n+2e-\frac{1}{e}}{e^2+2e-\frac{1}{e}}\right)$$

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{e^n}I\right)=$$

$$=\lim_{n\to\infty}\frac{1}{e^n}\left(-\frac{e^4}{(2e^2-1)^2}\cdot n+\frac{e}{(2e^2-1)}\left(1-\frac{1}{e^n}\right)+\frac{e^4}{(2e^2-1)^2}log\left(\frac{e^2e^n+2e-\frac{1}{e}}{e^2+2e-\frac{1}{e}}\right)\right)$$

$$=\lim_{n\to\infty}\frac{\frac{e^2e^n}{e^2e^n+2e-\frac{1}{e}}}{e^n}\frac{e^4}{(2e^2-1)^2}=\lim_{n\to\infty}\frac{1}{e^n}=0$$

Solution 3 by Serlea Kabay-Liberia

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{e^n}\int_0^n\frac{dx}{e^x\left(e^{x+2}+2e-\frac{1}{e}\right)}\right)=\lim_{n\to\infty}\left(\frac{1}{e^n(2e^2-1)}\int_0^n\left(\frac{e}{e^x}-\frac{e^3}{e^{x+2}+2e-\frac{1}{e}}\right)dx\right)$$

$$\overset{IBP}{=}\lim_{n\to\infty}\left(\frac{1}{e^n(2e^2-1)}\left(e-e^{1-n}-e^4\frac{n-log(1-2e^2-e^{n+3})}{2e^2-1}\right.\right.$$

$$\left.\left.+e^4\frac{log(1-2e^2-e^3)}{2e^2-1}\right)\right)=$$

$$=\lim_{n\to\infty}\frac{e-e^{1-n}}{e^n(2e^2-1)}-\lim_{n\to\infty}e^4\frac{n-log(1-2e^2-e^{n+3}}{e^n(2e^2-1)^2}-\lim_{n\to\infty}e^4\frac{log(1-2e^2-e^3)}{e^n(2e^2-1)^2}=0$$

Solution 4 by Akerele Olofin-Nigeria

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{e^n}\int_0^n\frac{dx}{e^x\left(e^{x+2}+2e-\frac{1}{e}\right)}\right);I=\int_0^n\frac{dx}{e^x\left(e^{x+2}+2e-\frac{1}{e}\right)}\overset{e^x=y}{=}$$

$$=\int_1^{e^n}\frac{1}{y\left(e^2y+2e-\frac{1}{e}\right)}\frac{dy}{y}=\int_1^{e^n}\frac{dy}{y^2\left(e^2y+2e-\frac{1}{e}\right)}=$$

$$=-\frac{e^4}{(2e^2-1)^2}\int_1^{e^n}\frac{1}{y}dy+\frac{e}{2e^2-1}\int_1^{e^n}\frac{1}{y^2}dy+\frac{e^4}{(2e^2-1)^2}\int_1^{e^n}\frac{e^2}{e^2y+2e-\frac{1}{e}}dy=$$

$$=-\frac{e^4}{(2e^2-1)^2}[logy]_1^{e^n}+\frac{e}{2e^2-1}\left[-\frac{1}{y}\right]_1^{e^n}+\frac{e^4}{(2e^2-1)^2}\left[log\left(e^2y+2e-\frac{1}{e}\right)\right]_1^{e^n}=$$

$$=-\frac{e^4}{(2e^2-1)^2}\cdot n+\frac{e}{(2e^2-1)}\left(1-\frac{1}{e^n}\right)+\frac{e^4}{(2e^2-1)^2}log\left(\frac{e^2e^n+2e-\frac{1}{e}}{e^2+2e-\frac{1}{e}}\right)$$

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{e^n}\int_0^n\frac{dx}{e^x\left(e^{x+2}+2e-\frac{1}{e}\right)}\right)=$$

$$=\lim_{n\to\infty}\frac{1}{e^n}\left(-\frac{e^4}{(2e^2-1)^2}\cdot n+\frac{e}{(2e^2-1)}\left(1-\frac{1}{e^n}\right)+\frac{e^4}{(2e^2-1)^2}log\left(\frac{e^2e^n+2e-\frac{1}{e}}{e^2+2e-\frac{1}{e}}\right)\right)$$

$$=\lim_{n\to\infty}\frac{\frac{e^2e^n}{e^2e^n+2e-\frac{1}{e}}}{e^n}\frac{e^4}{(2e^2-1)^2}=\lim_{n\to\infty}\frac{1}{e^n}=0$$

Solution 5 by Ravi Prakash-New Delhi-India

Let $f(x)=\frac{1}{e^x\left(e^{x+2}+2e-\frac{1}{e}\right)}, f'(x)=-\frac{\left(2e^{2x+2}+e^x\left(2e-\frac{1}{e}\right)\right)}{\left(e^{2x+2}+e^x\left(2e-\frac{1}{e}\right)\right)^2}<0, \forall x>0 \Rightarrow f$ –decreasing in

$$[0,n]\Rightarrow f(n)\le f(x)\le f(0), \forall x\in[0,n]\Leftrightarrow$$

$$f(n)\int_0^n dx\le\int_0^n f(x)dx\le f(0)\int_0^n dx\Leftrightarrow$$

$$\frac{n}{e^{2n}e^n\left(e^{n+2}+2e-\frac{1}{e}\right)}\le\frac{1}{e^n}\int_0^n f(x)dx\le\frac{n}{e^n\left(e^2+2e-\frac{1}{e}\right)}$$

$$\lim_{n\to\infty}\frac{n}{e^{2n}e^n\left(e^{n+2}+2e-\frac{1}{e}\right)}=\lim_{n\to\infty}\frac{n}{e^n\left(e^2+2e-\frac{1}{e}\right)}=0\Rightarrow$$

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{e^n}\int_0^n\frac{dx}{e^x\left(e^{x+2}+2e-\frac{1}{e}\right)}\right)=0$$

Solution 6 by Noor Alam-Kolkata-India

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{e^n}\int_0^n\frac{dx}{e^x\left(e^{x+2}+2e-\frac{1}{e}\right)}\right); I=\int_0^n\frac{dx}{e^x\left(e^{x+2}+2e-\frac{1}{e}\right)}\overset{A=\frac{2+\frac{1}{e}}{e^2}}{=}$$

$$=\frac{1}{e^2}\int_0^n\frac{dx}{e^x(e^x+A)}=\frac{1}{e^2}\int_0^n\frac{e^{-x}e^{-x}}{1+Ae^{-x}}dx\overset{1+Ae^{-x}}{=}\frac{1}{-Ae^2}\int_{1+A}^{1+Ae^{-n}}\frac{t-1}{At}dt=$$

$$=-\frac{1}{A^2e^2}\int_{1+A}^{1+Ae^{-n}}\left(1-\frac{1}{t}\right)dt=-\frac{1}{A^2e^2}[t-\log t]_{1+A}^{1+Ae^{-n}}=$$

$$=-\frac{1}{A^2e^2}\left(A(e^{-n}-1)+\log\left(\frac{1+A}{1+Ae^{-n}}\right)\right)$$

$$\Omega=\lim_{n\to\infty}\frac{1}{e^n}\cdot\left(-\frac{1}{A^2e^2}\left(A(e^{-n}-1)+\log\left(\frac{1+A}{1+Ae^{-n}}\right)\right)\right)=$$

$$=-\frac{1}{A^2e^2}\lim_{n\to\infty}\left[A(e^{-2n}-e^{-n})+\frac{1}{e^n}\log\left(\frac{1+A}{1+Ae^{-n}}\right)\right]=$$

$$= \frac{1}{A^2 e^2} \lim_{n\to\infty} log\left(\frac{1+Ae^{-n}}{Ae^{-n}}\right) \cdot \frac{Ae^{-n}}{e^n} = 0$$

Solution 7 by Timson Azeez Folorunsho-Nigeria

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{e^n}\int_0^n \frac{dx}{e^x\left(e^{x+2}+2e-\frac{1}{e}\right)}\right); I = \int_0^n \frac{dx}{e^x\left(e^{x+2}+2e-\frac{1}{e}\right)} \overset{e^x=y}{=}$$

$$= \int_1^{e^n} \frac{1}{y\left(e^2y+2e-\frac{1}{e}\right)}\frac{dy}{y} = \int_1^{e^n} \frac{dy}{y^2\left(e^2y+2e-\frac{1}{e}\right)} = I_1 + I_2 + I_3$$

$$I_1 = -\frac{e^4}{(2e^2-1)^2}\int_1^{e^n}\frac{1}{y}dy = -\frac{e^4}{(2e^2-1)^2}[logy]_1^{e^n} = -\frac{e^4}{(2e^2-1)^2}\cdot n$$

$$I_2 = \frac{e}{2e^2-1}\int_1^{e^n}\frac{1}{y^2}dy = \frac{e}{2e^2-1}\left[-\frac{1}{y}\right]_1^{e^n} = \frac{e}{(2e^2-1)}\left(1-\frac{1}{e^n}\right)$$

$$I_3 = \frac{e^4}{(2e^2-1)^2}\int_1^{e^n}\frac{e^2}{e^2y+2e-\frac{1}{e}}dy = \frac{e^4}{(2e^2-1)^2}\left[log\left(e^2y+2e-\frac{1}{e}\right)\right]_1^{e^n} =$$

$$= \frac{e^4}{(2e^2-1)^2} log\left(\frac{e^2e^n+2e-\frac{1}{e}}{e^2+2e-\frac{1}{e}}\right)$$

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{e^n}\int_0^n \frac{dx}{e^x\left(e^{x+2}+2e-\frac{1}{e}\right)}\right) =$$

$$= \lim_{n\to\infty}\frac{1}{e^n}\left(-\frac{e^4}{(2e^2-1)^2}\cdot n + \frac{e}{(2e^2-1)}\left(1-\frac{1}{e^n}\right) + \frac{e^4}{(2e^2-1)^2} log\left(\frac{e^2e^n+2e-\frac{1}{e}}{e^2+2e-\frac{1}{e}}\right)\right)$$

$$= \lim_{n\to\infty}\frac{\frac{e^2e^n}{e^2e^n+2e-\frac{1}{e}}}{e^n}\frac{e^4}{(2e^2-1)^2} = \lim_{n\to\infty}\frac{1}{e^n} = 0$$

Solution 8 by Remus Florin Stanca-Romania

$$\int \frac{dx}{e^x\left(e^{x+2}+2e-\frac{1}{e}\right)}=\int \frac{e^2dx}{e^{x+2}\left(e^{x+2}+2e-\frac{1}{e}\right)}=\frac{e^2}{2e-\frac{1}{e}}\int \frac{e^{x+2}+2e-\frac{1}{e}-e^{x+2}}{e^{x+2}\left(e^{x+2}+2e-\frac{1}{e}\right)}dx=$$

$$=\frac{e^3}{2e^2-1}\int\left(e^{-x-2}-\frac{1}{e^{x+2}+2e-\frac{1}{e}}\right)dx\,;(1)$$

$$\int e^{-x-2}dx=-e^{-x-2};(2)$$

$$\int \frac{dx}{e^{x+2}+2e-\frac{1}{e}}=\frac{1}{2e-\frac{1}{e}}\int \frac{2e-\frac{1}{e}}{e^{x+2}+2e-\frac{1}{e}}dx=\frac{e}{2e^2-1}\int \frac{e^{x+2}+2e-\frac{1}{e}-e^{x+2}}{e^{x+2}+2e-\frac{1}{e}}dx=$$

$$=\frac{e}{2e^2-1}\left(x-log\left(e^{x+2}+2e-\frac{1}{e}\right)\right)\Rightarrow$$

$$\int \frac{dx}{e^{x+2}+2e-\frac{1}{e}}=\frac{e}{2e^2-1}\left(x-log\left(e^{x+2}+2e-\frac{1}{e}\right)\right);(3)\overset{(1),(2),(3)}{\Longrightarrow}$$

$$\int \frac{dx}{e^x\left(e^{x+2}+2e-\frac{1}{e}\right)}=\frac{e^3}{2e^2-1}\left(e^{-x-2}-\frac{e}{2e^2-1}\left(x-log\left(e^{x+2}+2e-\frac{1}{e}\right)\right)\right)=$$

$$=-\frac{2e^{3-x}-e^{1-x}+e^4\left(x-log\left(e^{x+2}+2e-\frac{1}{e}\right)\right)}{(2e^2-1)^2}\Rightarrow$$

$$\int_0^n \frac{dx}{e^x\left(e^{x+2}+2e-\frac{1}{e}\right)}=$$

$$=-\frac{1}{(2e^2-1)^2}\left(2e^{3-n}-e^{1-n}+e^4\left(n-log\left(e^{n+2}+2e-\frac{1}{e}\right)\right)-2e^3+e+e^4log\left(e^2+2e-\frac{1}{e}\right)\right)$$

$$\lim_{n\to\infty}\frac{n}{e^n}=0,\ \lim_{n\to\infty}\frac{e^{3-n}}{e^n}=0$$

$$\lim_{n\to\infty}\frac{log\left(e^{n+2}+2e-\frac{1}{e}\right)}{e^n}=\lim_{n\to\infty}\frac{log\left(\frac{e^{n+3}+2e-\frac{1}{e}}{e^{n+2}+2e-\frac{1}{e}}\right)}{e^n(e-1)}$$

Therefore,

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{e^n}\int_0^n \frac{dx}{e^x\left(e^{x+2}+2e-\frac{1}{e}\right)}\right) = 0$$

1292. Find:

$$\Omega(n) = \int_{3-2\sqrt{2}}^{3+2\sqrt{2}} \frac{x^2 \log^{2n+1}x}{1+x^2}dx, n \in \mathbb{N}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohammad Rostami-Kabul-Afghanistan

$$\Omega(n) = \int_{3-2\sqrt{2}}^{3+2\sqrt{2}} \frac{x^2 \log^{2n+1}x}{1+x^2}dx = \int_{3-2\sqrt{2}}^{3+2\sqrt{2}} \frac{(x^2+1-1)\log^{2n+1}x}{1+x^2}dx =$$

$$= \int_{3-2\sqrt{2}}^{3+2\sqrt{2}} \log^{2n+1}x\, dx - \int_{3-2\sqrt{2}}^{3+2\sqrt{2}} \frac{\log^{2n+1}x}{1+x^2}dx = I_1 - I_2$$

$$I_2 = \int_{3-2\sqrt{2}}^{3+2\sqrt{2}} \frac{\log^{2n+1}x}{1+x^2}dx = \int_{3+2\sqrt{2}}^{3-2\sqrt{2}} \frac{-\log^{2n+1}x}{\frac{1+x^2}{x^2}}\left(-\frac{1}{x^2}dx\right) = -\int_{3-2\sqrt{2}}^{3+2\sqrt{2}} \frac{\log^{2n+1}x}{1+x^2}dx = -I_2$$

$$\Rightarrow I_2 = 0$$

$$\Omega(n) = \int_{3-2\sqrt{2}}^{3+2\sqrt{2}} \log^{2n+1}x\, dx = \int_{3-2\sqrt{2}}^{3+2\sqrt{2}} \frac{\partial^{2n+1}}{\partial a^{2n+1}}\bigg|_{a=0} x^a\, dx = \frac{\partial^{2n+1}}{\partial a^{2n+1}}\bigg|_{a=0} \int_{3-2\sqrt{2}}^{3+2\sqrt{2}} x^a\, dx =$$

$$= \frac{\partial^{2n+1}}{\partial a^{2n+1}}\bigg|_{a=0} \frac{x^{a+1}}{a+1}\bigg|_{3-2\sqrt{2}}^{3+2\sqrt{2}}$$

$$\Omega(n) = \frac{\partial^{2n+1}}{\partial a^{2n+1}}\bigg|_{a=0} \frac{\left(3+2\sqrt{2}\right)^{a+1} - \left(3-2\sqrt{2}\right)^{a+1}}{a+1}$$

Solution 2 by Mikael Bernardo-Mozambique

$$\Omega(n)=\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\frac{x^2\log^{2n+1}x}{1+x^2}dx=\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\log^{2n+1}x\,dx-\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\frac{\log^{2n+1}x}{1+x^2}dx$$

$$=\Omega_1(n)-\Omega_2(n)$$

$$\Omega_1(n)=\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\log^{2n+1}x\,dx\overset{\log x=y}{=}\int_{\log(3-2\sqrt{2})}^{\log(3+2\sqrt{2})}y^{2n+1}e^y dy\overset{IBP}{=}$$

$$=e^y y^{2n+1}\Big|_{\log(3-2\sqrt{2})}^{\log(3+2\sqrt{2})}-(2n+1)\int_{\log(3-2\sqrt{2})}^{\log(3+2\sqrt{2})}y^{2n}e^y dy=$$

$$=(3+2\sqrt{2})\log^{2n+1}(3+2\sqrt{2})-(3-2\sqrt{2})\log^{2n+1}(3-2\sqrt{2})-$$

$$-(2n+1)\left\{e^y y^{2n}\Big|_{\log(3-2\sqrt{2})}^{\log(3+2\sqrt{2})}-(2n)\int_{\log(3-2\sqrt{2})}^{\log(3+2\sqrt{2})}y^{2n-1}e^y dy\right\}=$$

$$=(3+2\sqrt{2})\log^{2n+1}(3+2\sqrt{2})-(3-2\sqrt{2})\log^{2n+1}(3-2\sqrt{2})-$$

$$-(2n+1)\{(3+2\sqrt{2})\log^{2n}(3+2\sqrt{2})-(3-2\sqrt{2})\log^{2n}(3-2\sqrt{2})\}+$$

$$+(2n)(2n+1)\{(3+2\sqrt{2})\log^{2n-1}(3+2\sqrt{2})-(3-2\sqrt{2})\log^{2n-1}(3-2\sqrt{2})\}-$$

$$-(2n+1)(2n)(2n-1)\left\{\dots(2n+1)!\int_{\log(3-2\sqrt{2})}^{\log(3+2\sqrt{2})}e^y dy\right\}=$$

$$=(3+2\sqrt{2})\log^{2n+1}(3+2\sqrt{2})-(3-2\sqrt{2})\log^{2n+1}(3-2\sqrt{2})-$$

$$-(2n+1)\{(3+2\sqrt{2})\log^{2n}(3+2\sqrt{2})-(3-2\sqrt{2})\log^{2n}(3-2\sqrt{2})\}+$$

$$+(2n)(2n+1)\{(3+2\sqrt{2})\log^{2n-1}(3+2\sqrt{2})-(3-2\sqrt{2})\log^{2n-1}(3-2\sqrt{2})\}-$$

$$-(2n+1)(2n)(2n-1)\{\dots(2n+1)!\,(4\sqrt{2})\}$$

$$\Omega_2(n)=\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\frac{\log^{2n+1}x}{1+x^2}dx\overset{\log x=y}{=}\int_{\log(3-2\sqrt{2})}^{\log(3+2\sqrt{2})}\frac{y^{2n+1}e^y}{1+e^{2y}}dy=$$

$$=\sum_{k=0}^{\infty}(-1)^k\int_{log(3-2\sqrt{2})}^{log(3+2\sqrt{2})}y^{2n+1}e^{y(2k+1)}dy\overset{IBP}{=}$$

$$=\sum_{k=0}^{\infty}(-1)^k\left\{\frac{(3+2\sqrt{2})log^{2n+1}(3+2\sqrt{2})-(3-2\sqrt{2})log^{2n+1}(3-2\sqrt{2})}{2k+1}-\frac{2n+1}{2k+1}\int_{log(3-2\sqrt{2})}^{log(3+2\sqrt{2})}y^{2n}e^{y(2k+1)}dy\right\}=$$

$$=\sum_{k=0}^{\infty}(-1)^k\left\{\frac{(3+2\sqrt{2})log^{2n+1}(3+2\sqrt{2})-(3-2\sqrt{2})log^{2n+1}(3-2\sqrt{2})}{2k+1}\right.$$

$$-(2n+1)\left\{\frac{(3+2\sqrt{2})log^{2n}(3+2\sqrt{2})-(3-2\sqrt{2})log^{2n}(3-2\sqrt{2})}{(2k+1)^2}\right\}$$

$$\left.+\frac{(2n+1)(2n)}{(2n+1)^3}\left[\dots-\frac{(2n+1)!}{(2k+1)^n}\int_{log(3-2\sqrt{2})}^{log(3+2\sqrt{2})}e^y dy\right]\right\}=$$

$$=\sum_{k=0}^{\infty}(-1)^k\left\{\frac{(3+2\sqrt{2})log^{2n+1}(3+2\sqrt{2})-(3-2\sqrt{2})log^{2n+1}(3-2\sqrt{2})}{2k+1}\right.$$

$$-(2n+1)\left\{\frac{(3+2\sqrt{2})log^{2n}(3+2\sqrt{2})-(3-2\sqrt{2})log^{2n}(3-2\sqrt{2})}{(2k+1)^2}\right\}$$

$$\left.+\frac{(2n+1)(2n)}{(2n+1)^3}\left[\dots-\frac{(2n+1)!}{(2k+1)^{n+1}}\left\{(3+2\sqrt{2})^{2k+1}-(3-2\sqrt{2})^{2k+1}\right\}\right]\right\}=$$

$$\Omega(n)=\Omega_1(n)-\Omega_2(n)$$

$$=(3+2\sqrt{2})log^{2n+1}(3+2\sqrt{2})-(3-2\sqrt{2})log^{2n+1}(3-2\sqrt{2})-$$

$$-(2n+1)\{(3+2\sqrt{2})log^{2n}(3+2\sqrt{2})-(3-2\sqrt{2})log^{2n}(3-2\sqrt{2})\}+$$

$$+(2n)(2n+1)\{(3+2\sqrt{2})log^{2n-1}(3+2\sqrt{2})-(3-2\sqrt{2})log^{2n-1}(3-2\sqrt{2})\}-$$

$$-(2n+1)(2n)(2n-1)\{\dots(2n+1)!\,(4\sqrt{2})\}-$$

$$-\sum_{k=0}^{\infty}(-1)^k\left\{\frac{(3+2\sqrt{2})log^{2n+1}(3+2\sqrt{2})-(3-2\sqrt{2})log^{2n+1}(3-2\sqrt{2})}{2k+1}\right.$$

$$-(2n+1)\left\{\frac{(3+2\sqrt{2})log^{2n}(3+2\sqrt{2})-(3-2\sqrt{2})log^{2n}(3-2\sqrt{2})}{(2k+1)^2}\right\}$$

$$\left.+\frac{(2n+1)(2n)}{(2n+1)^3}\left[\dots-\frac{(2n+1)!}{(2k+1)^{n+1}}\left\{(3+2\sqrt{2})^{2k+1}-(3-2\sqrt{2})^{2k+1}\right\}\right]\right\}$$

Solution 3 by Kaushik Mahanta-Assam-India

$$\Omega(n)=\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\frac{x^2\log^{2n+1}x}{1+x^2}dx=\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\frac{(x^2+1-1)\log^{2n+1}x}{1+x^2}dx=$$

$$=\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\log^{2n+1}x\,dx-\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\frac{\log^{2n+1}x}{1+x^2}dx=I_1-I_2$$

$$I_2=\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\frac{\log^{2n+1}x}{1+x^2}dx\overset{x=\frac{1}{y}}{=}\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\frac{-\log^{2n+1}y}{1+y^2}dy=-I_2\Rightarrow I_2=0$$

$$I_1=\frac{\partial^{2n+1}}{\partial s^{2n+1}}\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}x^s\,dx=\frac{\partial^{2n+1}}{\partial s^{2n+1}}\left(\frac{x^{s+1}}{s+1}\right)_{3-2\sqrt{2}}^{3+2\sqrt{2}}=$$

$$=\frac{\partial^{2n+1}}{\partial s^{2n+1}}\left(\frac{(3+2\sqrt{2})^{s+1}}{s+1}-\frac{(3-2\sqrt{2})^{s+1}}{s+1}\right)$$

$$\Omega(n)=\frac{\partial^{2n+1}}{\partial s^{2n+1}}\left(\frac{(3+2\sqrt{2})^{s+1}}{s+1}-\frac{(3-2\sqrt{2})^{s+1}}{s+1}\right)$$

1293. Find:

$$\Omega(a)=\int_0^a\left(\int_0^a\log(1+ax)^{\frac{1}{1+x^2}}dx\right)da\,,0<a<1$$

Proposed by Daniel Sitaru-Romania

Solution by Serlea Kabay-Liberia

$$\omega=\int_0^a\log(1+ax)^{\frac{1}{1+x^2}}dx=\int_0^a\frac{\log(1+ax)}{1+x^2}dx$$

$$\frac{d\omega}{da}=\int_0^a\frac{x}{(1+ax)(1+x^2)}dx+\frac{\log(1+a^2)}{1+a^2}=$$

$$=\frac{a}{a^2+1}\int_0^a\left(\frac{\frac{x}{a}+1}{x^2+1}-\frac{1}{ax+1}\right)dx+\frac{log(1+a^2)}{1+a^2}=$$

$$=\frac{atan^{-1}a}{a^2+1}+\frac{log(1+a^2)}{(1+a^2)}\Rightarrow\omega=\int_0^a\left(\frac{atan^{-1}a}{a^2+1}+\frac{log(1+a^2)}{(1+a^2)}\right)da=I_1+I_2,$$

$$I_1=\int_0^a\frac{log(1+a^2)}{1+a^2}da\,;I_2=\int_0^a\frac{atan^{-1}a}{1+a^2}da$$

Using IBP, we get:

$$I=log(1+a^2)tan^{-1}x-2I_2\Rightarrow\omega=I_2+\frac{log(1+a^2)tan^{-1}x-2I_2}{2}$$

$$\Rightarrow\omega=\frac{log(1+a^2)tan^{-1}x}{2}$$

$$\Omega(a)=\int_0^a\left(\int_0^a log(1+ax)^{\frac{1}{1+x^2}}dx\right)da=\int_0^a\frac{log(1+a^2)tan^{-1}x}{2}da\overset{IBP}{=}$$

$$=\frac{1}{2}log(1+a^2)\left(atan^{-1}a-\frac{1}{2}log(1+x^2)\right)-\int_0^a\left(\frac{a^2tan^{-1}a}{a^2+1}+\frac{log(1+a^2)}{2(1+a^2)}\right)da$$

By separation:

$$I_3=\int_0^a\frac{a^2tan^{-1}a}{1+a^2}da\,,I_4=\int_0^a\frac{alog(1+a^2)}{2(1+a^2)}da$$

$$I_3=\int_0^a\frac{a^2tan^{-1}a}{1+a^2}da\overset{IBP}{=}atan^{-1}a-\frac{1}{2}log(1+a^2)-\left(tan^{-1}a\right)^2$$

$$I_4=\int_0^a\frac{alog(1+a^2)}{2(1+a^2)}da=\frac{1}{8}log^2(1+x^2)$$

$$\Omega(a)=\frac{1}{2}(tan^{-1}a)^2-atan^{-1}a+\frac{a}{2}log(1+a^2)tan^{-1}a+\frac{1}{2}log(1+a^2)-$$

$$-\frac{1}{8}log^2(1+a^2)$$

Let $I=\int_0^a\frac{log(1+ax)}{1+x^2}dx\Rightarrow\frac{\partial I}{\partial a}=\int_0^a\frac{x}{(1+ax)(1+x^2)}dx+\frac{log(1+a^2)}{1+a^2}$

$$\Omega_1 = \int_0^a \frac{x}{(1+ax)(1+x^2)}dx = \left(\frac{1}{1+a^2}\right)\int_0^a \left(\frac{x+a}{1+x^2} - \frac{a}{1+ax}\right)dx =$$

$$= \frac{1}{1+a^2}\left[log\sqrt{1+a^2} + a \cdot tan^{-1}(a) - log(1+a^2)\right]$$

$$\frac{\partial I}{\partial a} = \frac{1}{1+a^2}\left[log\sqrt{1+a^2} + a \cdot tan^{-1}(a) - log(1+a^2)\right] + \frac{log(1+a^2)}{1+a^2} =$$

$$= \frac{1}{1+a^2}\left[\frac{1}{2}log(1+a^2) + a \cdot tan^{-1}(a)\right]$$

$$I_3 = \frac{log(1+a^2)tan^{-1}(a)}{2}$$

$$\Omega(a) = \int_0^a \left(\frac{log(1+a^2)}{2(1+a^2)} + \frac{atan^{-1}(a)}{1+a^2}\right)da = \int_0^a \frac{log(1+a^2)}{2(1+a^2)}da + \int_0^a \frac{atan^{-1}(a)}{1+a^2}da$$

$$I_3 = \int_0^a \frac{log(1+a^2)}{2(1+a^2)}da, I_4 = \int_0^a \frac{atan^{-1}(a)}{1+a^2}da \overset{IBP}{\Longrightarrow}$$

$$I_3 = \frac{log(1+a^2)tan^{-1}(a)}{2} - I_4$$

Therefore,

$$\Omega(a) = \int_0^a \left(\int_0^a log(1+ax)^{\frac{1}{1+x^2}}dx\right)da = \frac{log(1+a^2)tan^{-1}(a)}{2} - I_4 + I_4 =$$

$$= \frac{log(1+a^2)tan^{-1}(a)}{2}$$

1294. If $0 < a \le b$ then find:

$$\Omega = \int_a^b log\left(\frac{\left(1+\frac{x}{a}\right)^{x^{-1}\cdot e^{\frac{b}{x}}}}{\left(1+\frac{b}{x}\right)^{x^{-1}\cdot e^{\frac{x}{a}}}}\right)dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$\Omega=\int_a^b \log\left(\frac{\left(1+\frac{x}{a}\right)^{x^{-1}\cdot e^{\frac{b}{x}}}}{\left(1+\frac{b}{x}\right)^{x^{-1}\cdot e^{\frac{x}{a}}}}\right)dx=\int_a^b \frac{e^{\frac{b}{x}}}{x}\log\left(1+\frac{x}{a}\right)dx-\int_a^b \frac{e^{\frac{x}{a}}}{x}\log\left(1+\frac{b}{x}\right)dx=I_1-I_2$$

$$I_1=\int_a^b \frac{e^{\frac{b}{x}}}{x}\log\left(1+\frac{x}{a}\right)dx \overset{x=\frac{ab}{y}}{=} \int_a^b \frac{e^{\frac{y}{a}}}{y}\log\left(1+\frac{b}{y}\right)dy=I_2 \Rightarrow \Omega=0.$$

Solution 2 by Ali Jaffal-Lebanon

$$\text{Let } f(x)=\log\left(\frac{\left(1+\frac{x}{a}\right)^{x^{-1}\cdot e^{\frac{b}{x}}}}{\left(1+\frac{x}{b}\right)^{x^{-1}\cdot e^{\frac{x}{a}}}}\right)\Rightarrow f\left(\frac{ab}{x}\right)=\log\left(\frac{\left(1+\frac{b}{x}\right)^{\frac{x}{ab}e^{\frac{x}{a}}}}{\left(1+\frac{x}{a}\right)^{\frac{x}{ab}e^{\frac{b}{x}}}}\right)^{\frac{x^2}{ab}}=-\frac{x^2}{ab}f(x)$$

$$\Omega=\int_a^b f(x)dx \overset{t=\frac{ab}{x}}{=} \int_a^b f\left(\frac{ab}{t}\right)\frac{ab}{t^2}dt=-\int_a^b \frac{t^2}{ab}f(t)\cdot\frac{ab}{t^2}dt=-\int_a^b f(t)dt=-\Omega\Rightarrow\Omega=0$$

Solution 3 by Kaushik Mahanta-Assam-India

$$\Omega=\int_a^b \log\left(\frac{\left(1+\frac{x}{a}\right)^{x^{-1}\cdot e^{\frac{b}{x}}}}{\left(1+\frac{b}{x}\right)^{x^{-1}\cdot e^{\frac{x}{a}}}}\right)dx=\int_a^b \frac{e^{\frac{b}{x}}}{x}\log\left(1+\frac{x}{a}\right)dx-\int_a^b \frac{e^{\frac{x}{a}}}{x}\log\left(1+\frac{b}{x}\right)dx=I_1-I_2$$

$$I_2=\int_a^b \frac{e^{\frac{x}{a}}}{x}\log\left(1+\frac{b}{x}\right)dx \overset{\frac{b}{x}=\frac{t}{a}}{=} \int_a^b \frac{e^{\frac{b}{t}}}{ab}\cdot t\log\left(1+\frac{t}{a}\right)\left(\frac{ab}{t^2}\right)dt=\int_a^b \frac{e^{\frac{b}{t}}}{t}\log\left(1+\frac{t}{a}\right)dt=I_1$$

$$\Rightarrow\Omega=0$$

1295. Find without any software:

$$\Omega=\int_0^\infty \frac{(coshx-sinhx)(4sinh^2x-sinh2x+2)}{(coshx+sinhx)(4sinh^2x+sinh2x+2)}dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohammad Rostami-Kabul-Afghanistan

$$\frac{coshx - sinhx}{coshx + sinhx} = \frac{\frac{e^x + e^{-x} - e^x + e^{-x}}{2}}{\frac{e^x + e^{-x} + e^x - e^{-x}}{2}} = e^{-2x}$$

$$\frac{4sinh^2x - sinh2x + 2}{4sinh^2x + sinhx + 2} = \frac{4sinh^2x - sinh2x + 2cosh^2x - 2sinh^2x}{4sinh^2x + sinh2x + 2cosh^2x - 2sinh^2x} =$$

$$= \frac{2cosh2x - sinh2x}{2cosh2x + sinh2x} = \frac{e^{2x} + 3e^{-2x}}{e^{-2x} + 3e^{2x}}$$

$$\Omega = \int_0^\infty e^{-2x}\left(\frac{e^{2x} + 3e^{-2x}}{e^{-2x} + 3e^{2x}}\right)dx = \int_0^\infty \frac{1 + 3e^{-4x}}{e^{-2x} + 3e^{2x}}dx = \int_0^\infty \frac{e^{-2x} + 3e^{-6x}}{e^{-4x} + 3}dx \overset{e^{-x}=t}{=}$$

$$= \int_0^1 \frac{t(3t^4 + 1)}{t^4 + 3}dt = \int_0^1 \frac{t(3(t^4 + 3 - 3) + 1)}{t^4 + 3}dt = \int_0^1 \frac{t(3(t^4 + 3) - 8)}{t^4 + 3}dt =$$

$$= \int_0^1 3t\,dt - \int_0^1 \frac{8t}{t^4 + 3}dt = \frac{3}{2} - 4\int_0^1 \frac{2t}{(t^2)^2 + 3}dt \overset{t^2=u}{=} \frac{3}{2} - 4\int_0^1 \frac{du}{u^2 + (\sqrt{3})^2} = \frac{3}{2} - \frac{2\pi}{3\sqrt{3}}$$

Solution 2 by Yen Tung Chung-Taichung-Taiwan

$$\Omega = \int_0^\infty \frac{(coshx - sinhx)(4sinh^2x - sinh2x + 2)}{(coshx + sinhx)(4sinh^2x + sinh2x + 2)}dx =$$

$$= \int_0^\infty \frac{e^{-x}\left(4\left(\frac{e^x - e^{-x}}{2}\right)^2 - \frac{e^{2x} - e^{-2x}}{2} + 2\right)}{e^x\left(4\left(\frac{e^x - e^{-x}}{2}\right)^2 + \frac{e^{2x} - e^{-2x}}{2} + 2\right)}dx = \int_0^\infty \frac{e^{4x} + 3}{e^{2x}(3e^{4x} + 1)}dx =$$

$$= \int_0^\infty \frac{1 + 3e^{-4x}}{3 + e^{-4x}}e^{-2x}dx \overset{y=e^{-2x}}{=} \frac{1}{2}\int_0^1 \frac{3y^2 + 1}{y^2 + 3}dy = \frac{1}{2}\int_0^1 \left(3 - \frac{8}{y^2 + 3}\right)dy =$$

$$= \frac{1}{2}\left(3y - \frac{8}{\sqrt{3}}tan^{-1}\left(\frac{y}{\sqrt{3}}\right)\right)\Bigg|_0^1 = \frac{3}{2} - \frac{2\pi}{3\sqrt{3}}$$

Solution 3 by Mikael Bernardo-Mozambique

$$\Omega = \int_0^\infty \frac{(\cosh x - \sinh x)(4\sinh^2 x - \sinh 2x + 2)}{(\cosh x + \sinh x)(4\sinh^2 x + \sinh 2x + 2)} dx =$$

$$= \int_0^\infty \frac{(\cosh x - \sinh x)(2\cosh^2 x - \sinh x\cosh x + 1)}{(\cosh x + \sinh x)(2\sinh^2 x + \sinh x\cosh x + 1)} dx =$$

$$= \int_0^\infty \frac{\left(\frac{1-\tanh x}{\operatorname{sech} x}\right)\left(2\frac{\tanh^2 x}{\operatorname{sech}^2 x} - \frac{\tanh x}{\operatorname{sech}^2 x} + 1\right)}{\left(\frac{1+\tanh x}{\operatorname{sech} x}\right)\left(2\frac{\tanh^2 x}{\operatorname{sech}^2 x} + \frac{\tanh x}{\operatorname{sech}^2 x} + 1\right)} dx =$$

$$= \int_0^\infty \frac{(1+\tanh^2 x)\left(\frac{2\tanh^2 x - \tanh x + 1 - \tanh^2 x}{\operatorname{sech}^2 x}\right)}{(1+\tanh x)^2\left(\frac{2\tanh^2 x - \tanh x + 1 - \tanh^2 x}{\operatorname{sech}^2 x}\right)} dx =$$

$$= \int_0^\infty \frac{\operatorname{sech}^2 x(\tanh^2 x - \tanh x + 1)}{(1 + 2\tanh x + \tanh^2 x)(\tanh^2 x + \tanh x + 1)} dx =$$

$$= \int_0^\infty \frac{\operatorname{sech}^2 x(\tanh^2 x - \tanh x + 1)dx}{(\tanh^2 x + \tanh x + 1 + 2\tanh^3 x + 2\tanh^2 x + 2\tanh x + \tanh^4 x + \tanh^3 x + \tanh^2 x)}$$

$$= \int_0^\infty \frac{\operatorname{sech}^2 x(\tanh^2 x - \tanh x + 1)}{\tanh^4 x + 3\tanh^3 x + 4\tanh^2 x + 3\tanh x + 1} dx \overset{u=\tanh x}{=} \int_0^1 \frac{u^2 - u + 1}{u^4 + 3u^3 + 4u^2 + 3u + 2} du$$

$$= \int_0^1 \left(\frac{3}{(u+1)^2} - \frac{2}{u^2 + u + 1}\right) du = -\frac{3}{u+1}\Big|_0^1 - 2\int_0^1 \frac{du}{\left(u+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} =$$

$$= \frac{3}{2} - \frac{4}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(u + \frac{1}{2}\right)\right)\right]_0^1 = \frac{3}{2} - \frac{2\pi}{3\sqrt{3}}$$

1296. Find without any software:

$$\Omega = \int \frac{x^{2018}}{1 - x^{4038}} dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Kaushik Mahanta-Assam-India

$$\Omega=\int\frac{x^{2018}}{1-x^{4038}}dx=\int\frac{x^{2018}}{1+(x^{2019})^2}dx\overset{x^{2019}=t}{=}\frac{1}{2019}\int\frac{dt}{1-t^2}=$$

$$=\frac{1}{2\cdot 2019}log\left|\frac{1+t}{1-t}\right|+C=\frac{1}{4038}log\left|\frac{1+x^{2019}}{1-x^{2019}}\right|+C$$

Solution 2 by Eldeniz Hesenov-Georgia

$$\Omega=\int\frac{x^{2018}}{1-x^{4038}}dx=\int\frac{x^{2018}}{1+(x^{2019})^2}dx\overset{x^{2019}=u}{=}\frac{1}{2019}\int\frac{du}{1-u^2}=$$

$$=\frac{1}{2\cdot 2019}log\left|\frac{1+u}{1-u}\right|+C=\frac{1}{4038}log\left|\frac{1+x^{2019}}{1-x^{2019}}\right|+C$$

Solution 3 by Pranesh Pyara Shretsha-Nepal

$$\Omega=\int\frac{x^{2018}}{1-x^{4038}}dx=\int\frac{x^{2018}}{1+(x^{2019})^2}dx\overset{x^{2019}=u}{=}\frac{1}{2019}\int\frac{du}{1-u^2}=$$

$$=\frac{1}{2\cdot 2019}log\left|\frac{1+u}{1-u}\right|+C=\frac{1}{4038}log\left|\frac{1+x^{2019}}{1-x^{2019}}\right|+C$$

Solution 4 by Hafiz Iqbal-Situbondo-Indonesia

$$\Omega=\int\frac{x^{2018}}{1-x^{4038}}dx=-\int\frac{x^{2018}}{x^{4038}-1}dx\overset{u=x^{2019}}{=}-\frac{1}{2019}\int\frac{du}{u^2-1}=$$

$$=-\frac{1}{2019}\left(\frac{1}{2}\int\frac{du}{u-1}-\frac{1}{2}\int\frac{du}{u+1}\right)=-\frac{1}{2019}\frac{1}{2}(log|u-1|-log|u+1|)=$$

$$=\frac{1}{4038}log\left|\frac{1+x^{2019}}{1-x^{2019}}\right|+C$$

Solution 5 by Probal Chakraborty-India

$$\Omega=\int\frac{x^{2018}}{1-x^{4038}}dx=\int\frac{x^{2018}}{1+(x^{2019})^2}dx\overset{x^{2019}=cost}{=}-\frac{1}{2019}\int\frac{sint}{1-cos^2t}dt=$$

$$=-\frac{1}{2019}\int\frac{dt}{sint}=-\frac{1}{2019}\int csc\,t\,dt=-\frac{1}{2019}log|csc\,t-cot\,t|+C=$$

$$=\frac{1}{2019}log|csc\,t+cot\,t|+C=\frac{1}{2019}log\left|\frac{1+cos\,t}{sin\,t}\right|=\frac{1}{4038}log\left|\frac{1+x^{2019}}{1-x^{2019}}\right|+C$$

Solution 6 by Arslan Ahmed-Yemen

$$\Omega=\int\frac{x^{2018}}{1-x^{4038}}dx=\int\frac{x^{2018}}{1+(x^{2019})^2}dx=\frac{1}{2019}\int\frac{2019x^{2018}}{1+(x^{2019})^2}dx=$$

$$=\frac{1}{2019}tanh^{-1}(x^{2019})+C$$

Solution 7 by Timson Azeez Folorunsho-Nigeria

$$\Omega=\int\frac{x^{2018}}{1-x^{4038}}dx=\int\frac{x^{2018}}{1+(x^{2019})^2}dx\overset{x^{2019}=y}{=}\frac{1}{2019}\int\frac{dy}{1-y^2}=$$

$$=\frac{1}{2019}\int\left(\frac{1}{2(1-y)}+\frac{1}{2(1+y)}\right)dy=\frac{1}{4038}log|1+y|-\frac{1}{4038}log|1-y|=$$

$$=\frac{1}{2\cdot 2019}log\left|\frac{1+u}{1-u}\right|+C=\frac{1}{4038}log\left|\frac{1+x^{2019}}{1-x^{2019}}\right|+C$$

1297. If $n>0$, then prove:

$$\int_0^\infty\frac{log(coshx)}{cosh^n x}dx=\int_0^\infty\frac{\psi(n)-\psi\left(\frac{n}{2}\right)-log2}{cosh^n x}dx$$

-where $\psi(n)$ is the digamma function

Proposed by Angad Singh-Pune-India

Solution 1 by Rana Ranino-Setif-Algerie

$$I(n)=\int_0^\infty\frac{dx}{cosh^n x}\overset{u=coshx}{=}\int_1^\infty\frac{du}{u^n\cdot\sqrt{u^2-1}}\overset{t=\frac{1}{u^2}}{=}\frac{1}{2}\int_0^1 t^{\frac{n}{2}-1}(1-t)^{-\frac{1}{2}}dt=\frac{1}{2}B\left(\frac{n}{2},\frac{1}{2}\right)$$

$$\Omega=\int_0^\infty\frac{log(coshx)}{cosh^n x}dx=\frac{\partial}{\partial s}\bigg|_{s=0}\int_0^\infty\frac{dx}{(coshx)^{n-s}}=\frac{1}{2}\cdot\frac{\partial}{\partial s}\bigg|_{s=0}I(n-s)=$$

$$=\frac{1}{2}\cdot\frac{\partial}{\partial f}\bigg|_{s=0}B\left(\frac{n-s}{2},\frac{1}{2}\right)$$

$$\Omega=\frac{1}{2}\left[B\left(\frac{n+s}{2},\frac{1}{2}\right)\left\{\frac{\psi^{(0)}\left(\frac{n-s+1}{2}\right)-\psi^{(0)}\left(\frac{n-s}{2}\right)}{2}\right\}\right]_{s=0}=$$

$$=\frac{1}{2}B\left(\frac{n}{2},\frac{1}{2}\right)\left(\frac{\psi^{(0)}\left(\frac{n+1}{2}\right)-\psi^{(0)}\left(\frac{n}{2}\right)}{2}\right)$$

$$\Omega=\underbrace{\frac{1}{2}B\left(\frac{n}{2},\frac{1}{2}\right)}_{I(n)}\left\{\psi^{(0)}(n)-\psi^{(0)}\left(\frac{n}{2}\right)-log2\right\}=\int_0^{\infty}\frac{\psi^{(0)}(n)-\psi^{(0)}\left(\frac{n}{2}\right)-log2}{cosh^n x}dx$$

Therefore,

$$\int_0^{\infty}\frac{log(coshx)}{cosh^n x}dx=\int_0^{\infty}\frac{\psi(n)-\psi\left(\frac{n}{2}\right)-log2}{cosh^n x}dx$$

Solution 2 by Akerele Olofin-Nigeria

$$\Omega=\int_0^{\infty}\frac{log(coshx)}{cosh^n x}dx=\frac{\partial}{\partial a}\bigg|_{a=0}\int_0^{\infty}cosh^{a-n}x\,dx=$$

$$\Omega=\frac{1}{2}\cdot\frac{\partial}{\partial a}\bigg|_{a=0}\frac{\Gamma\left(\frac{n-a}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n-a+1}{2}\right)}=\frac{1}{2}\beta\left(\frac{n}{2},\frac{1}{2}\right)\left\{\frac{\psi_o\left(\frac{n+1}{2}\right)-\psi_o\left(\frac{n}{2}\right)}{2}\right\}=$$

$$=\frac{1}{2}\beta\left(\frac{n}{2},\frac{1}{2}\right)\left\{\psi_o(n)-\psi_o\left(\frac{n}{2}\right)-log2\right\}$$

Therefore,

$$\int_0^{\infty}\frac{log(coshx)}{cosh^n x}dx=\int_0^{\infty}\frac{\psi(n)-\psi\left(\frac{n}{2}\right)-log2}{cosh^n x}dx$$

1298. **Define the system of differential as:**

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&-1&-1&-1&0\end{bmatrix}$$

With the following initial conditions: $x_1(0)=0, x_2(0)=1,$

$x_3(0)=2, x_4(0)=3.$

Then show that a single function $x=f(t)$ **which also satisfied the above system is** $x(t)=-2e^{-t}-\sin t-4\cos t+6$

Proposed by Tobi Joshua-Nigeria

Solution by proposer

We have to convert it to a single higher order differential equation and show that the solution of such differential equation satisfies from the above matrix, we can write:

$$\begin{cases} x_1' = x_2; (1) \\ x_2' = x_3; (2) \\ x_3' = x_4; (3) \\ x_4' = x_5; (4) \\ x_5' = -x_2 - x_3 - x_4; (5) \end{cases}$$

WLOG, defined $x(0) = x_1(0) = 0, x'(0) = x_2(0) = 1, x''(0) = x_3(0) = 2,$

$$x''(0) = x_4(0) = 3; (6) \overset{(5)}{\Rightarrow} x^{(iv)} = -x' - x'' - x''' \to x^{(iv)} + x''' + x'' + x' = 0$$

$x(0) = 0, x'(0) = 1, x''(0) = 2, x'''(0) = 3$, **we have a fourth order differential equation.**

Characteristic equation is: $m^4 + m^3 + m^2 + m = 0 \to$

$$m(m+1)(m^2+1) = 0 \to m = 0, m = -1, m = \pm i$$

$$x(t) = (c_1 cost + c_2 sint) + c_3 + c_4 e^{-t}; x(0) = 0 \to c_1 + c_3 + c_4 = 0; (7)$$

$$x'(0) = 1, c_2 - c_4 = 1; (8), x''(0) = 2, -c_1 + c_4 = 2; (9),$$

$$x'''(0) = 3, -c_2 - c_4 = 3; (10)$$

Solving equation $(1) - (10) \to c_1 = -4, c_2 = -1, c_3 = 6, c_4 = -2.$

There $x = f(t) = (-4cost - sint) + 6 - 2e^{-t}$ **is the only function satisfying the system.**

1299.

$$\frac{\mathcal{F}_x[sin^2(e^{-x})](a)}{\mathcal{F}_x\left[\left(e^{-x}sin(e^{-x})\right)^2\right](a)} = \frac{4}{a^2 + ia}$$

$\mathcal{F}_x[\cdot](a)$ **−is Fourier transform.**

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Akerele Olofin-Nigeria

$$\mathcal{F}_x[sin^2(e^{-x})](a) = \Omega \Rightarrow \Omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} sin^2(e^{-x}) e^{-iax} dx \overset{t=e^{-x}}{=}$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{e^{it}-e^{-it}}{2i}\right)^2 t^{ia-1}\,dt=-\frac{1}{4\sqrt{2\pi}}\left(\int_0^{\infty} t^{ia-1}e^{2it}-2t^{ia-1}+e^{-2it}t^{ia-1}dt\right)$$

Let $\varphi=-\frac{1}{4\sqrt{2\pi}}$ then

$$\Omega=\varphi\int_0^{\infty} t^{ia-1}e^{-\frac{2t}{i}}dt-2\varphi\int_0^{\infty} t^{-\frac{a}{i}-1}\,dt+\varphi\int_0^{\infty} t^{ia-1}e^{-2it}dt$$

$$\mathcal{F}_x[e^{-2x}sin^2(e^{-x})](a)=\Lambda$$

$$\Lambda=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2x}e^{-2aix}sin^2(e^{-x})dx\overset{t=e^{-x}}{=}\frac{1}{\sqrt{2\pi}}\int_0^{\infty} t^{ia+1}sin^2t\,dt=$$

$$=-\frac{1}{4\sqrt{2\pi}}\int_0^{\infty} t^{ia+1}(e^{2it}-2+e^{-2it})\,dt=$$

$$=\varphi\left(\int_0^{\infty} t^{ai+1}e^{-\frac{2}{i}t}\,dt+\int_0^{\infty} t^{ai+1}e^{-2it}\,dt\right)=\varphi\left(\frac{\Gamma(ia+2)}{\left(\frac{2}{i}\right)^{ia+2}}+\frac{\Gamma(ia+2)}{(2i)^{ia+2}}\right)=$$

$$=\varphi\Gamma(ia+2)\left(\frac{-e^{-\pi a}-1}{2^{ia+2}e^{-\frac{\pi a}{2}}}\right)$$

Hence,

$$\frac{\mathcal{F}_x[sin^2(e^{-x})](a)}{\mathcal{F}_x\left[\left(e^{-x}sin(e^{-x})\right)^2\right](a)}=\frac{\Omega}{\Lambda}=\frac{\Gamma(ia)}{\Gamma(ia+2)}\frac{4(e^{-\pi a}+1)}{2^{ia}e^{-\frac{\pi a}{2}}}\frac{2^{ia}e^{-\frac{\pi a}{2}}}{-e^{-\pi a}-1}=\frac{4}{a^2+ia}$$

Therefore,

$$\frac{\mathcal{F}_x[sin^2(e^{-x})](a)}{\mathcal{F}_x\left[\left(e^{-x}sin(e^{-x})\right)^2\right](a)}=\frac{4}{a^2+ia}$$

Solution 2 by Probal Chakraborty-India

$$\mathcal{F}_x[sin^2(e^{-x})](a)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} sin^2(e^{-x})e^{-iax}dx\overset{t=e^{-x}}{=}\frac{1}{\sqrt{2\pi}}\int_0^{1} t^{ia}sin^2t\,dt$$

$$\mathcal{F}_x\left[\left(e^{-x}sin(e^{-x})\right)^2\right](a)=\frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-2x}e^{-iax}sin^2(e^{-x})\,dx=\frac{1}{\sqrt{2\pi}}\int_0^{1} t^{2+ia}sin^2t\,dt$$

$$\frac{\mathcal{F}_x[\sin^2(e^{-x})](a)}{\mathcal{F}_x\left[\left(e^{-x}\sin(e^{-x})\right)^2\right](a)}=\frac{\int_0^1 t^{ia}\sin^2 t\,dt}{\int_0^1 t^{2+ia}\sin^2 t\,dt}=$$

$$=\frac{4\,{}_1F_2\left(\frac{ia}{2}+\frac{3}{2};\frac{3}{2};\frac{ia}{2}+\frac{5}{2};-1\right)+(3+ia)(\cos 2-1)}{-6+2\{ia(ia-4i)\}}$$

$$\cdot\frac{-6+2(ia+2)(ia+2-4i)}{4\,{}_1F_2\left(\frac{ia}{2}+1+\frac{3}{2};\frac{3}{2};\frac{5}{2}+\frac{ia}{2}+1;-1\right)+(5+ia)(\cos 2-1)}=$$

$$=\frac{\left[4\,{}_1F_2\left(\frac{ia}{2}+\frac{3}{2};\frac{3}{2};\frac{ia}{2}+\frac{5}{2};-1\right)+(3+ia)(\cos 2-1)\right]\cdot[-6+2(ia+2)(ia+2-4i)]}{[-6+2\{ia(ia-4i)\}]\cdot\left[4\,{}_1F_2\left(\frac{ia}{2}+1+\frac{3}{2};\frac{3}{2};\frac{5}{2}+\frac{ia}{2}+1;-1\right)+(5+ia)(\cos 2-1)\right]}$$

$$=\frac{4}{a^2+ia}$$

Therefore,

$$\frac{\mathcal{F}_x[\sin^2(e^{-x})](a)}{\mathcal{F}_x\left[\left(e^{-x}\sin(e^{-x})\right)^2\right](a)}=\frac{4}{a^2+ia}$$

1300. $\sigma(x)=\Gamma(x)^{\Gamma(x)^{\Gamma(x)^{\cdots}}}$

Show that $\sigma(3)=\frac{-w(-\log 2)}{\log 2}$

where $w(\cdot)$ **—denotes the Lambert** w **function.**

Proposed by Akerele Olofin-Nigeria

Solution by Arslan Ahmed-Yemen

$$\sigma(x)=\Gamma(x)^{\Gamma(x)^{\Gamma(x)^{\cdots}}}\Rightarrow\Gamma(x)^{\sigma(x)}=\sigma(x)\Rightarrow e^{\sigma(x)\log(\Gamma(x))}=\sigma(x)\Rightarrow$$

$$\left[e^{\sigma(x)\log(\Gamma(x))}\right]^{-1}=\frac{1}{\sigma(x)}\Rightarrow e^{-\sigma(x)\log(\Gamma(x))}=\frac{1}{\sigma(x)}$$

$$\Rightarrow-\sigma(x)\log(\Gamma(x))e^{-\sigma(x)\log(\Gamma(x))}=\frac{-\sigma(x)\log(\Gamma(x))}{\sigma(x)}$$

$$w\left(-\sigma(x)\log(\Gamma(x))e^{-\sigma(x)\log(\Gamma(x))}\right)=w(-\log(\Gamma(x)))\Rightarrow$$

$$\sigma(3)=\frac{-w\left(-\log(\Gamma(3))\right)}{\log(\Gamma(3))}\Rightarrow\sigma(3)=\frac{-w(-\log 2)}{\log 2}$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru