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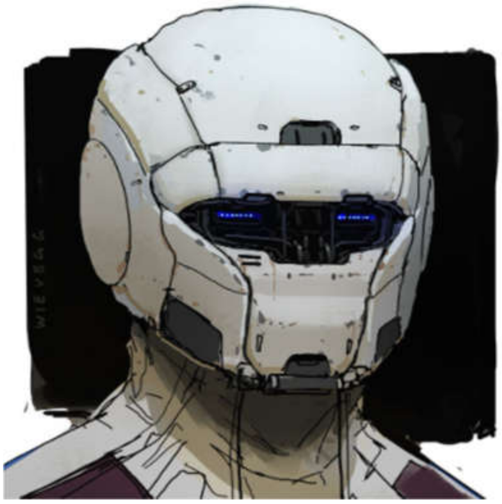

Prove that:

$$\lim_{n\to\infty}\sum_{k=0}^{n} tan^{-1}\left(\frac{n}{n^2+k^2+k}\right)=\frac{\pi}{4}$$

Proposed by Mohammed Bouras-Fes-Morocco

Solution 1 by Samar Das-India, Solution 2 by Khaled Abd Imouti-Damascus-Syria, Solution 3 by Mohammad Rostami-Kabul-Afghanistan

Solution 1 by Samar Das-India

$$\lim_{n\to\infty}\sum_{k=0}^{n} tan^{-1}\left(\frac{n}{n^2+k^2+k}\right)=\lim_{n\to\infty}\sum_{k=0}^{n} tan^{-1}\left(\frac{n}{n^2+k(k+1)}\right)=$$

$$=\lim_{n\to\infty}\sum_{k=0}^{n} tan^{-1}\left(\frac{\frac{1}{n}}{\frac{k+1}{n}\cdot\frac{k}{n}+1}\right)=\lim_{n\to\infty}\sum_{k=0}^{n} tan^{-1}\left(\frac{\frac{k+1}{n}-\frac{k}{n}}{1+\frac{k+1}{n}\cdot\frac{k}{n}}\right)=$$

$$=\lim_{n\to\infty}\sum_{k=0}^{n}\left(tan^{-1}\left(\frac{k+1}{n}\right)-tan^{-1}\left(\frac{k}{n}\right)\right)=\lim_{n\to\infty} tan^{-1}\left(1+\frac{1}{n}\right)=\frac{\pi}{4}$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

If $(\alpha,\beta)\in\mathbb{R}_+^*\times\mathbb{R}$ and $\gamma=\alpha^2+\beta^2+\beta$ then:

$$tan^{-1}\left(\frac{x+\beta+1}{\alpha}\right)+tan^{-1}\left(\frac{x+\beta}{\alpha}\right)=tan^{-1}\left(\frac{\alpha}{x^2+(2\beta+1)x+\gamma}\right)$$

For $\beta=0, x=k$ then $\gamma=k^2>0$. So, suppose $k=\alpha,\alpha=n,\beta=0$, we have:

$$tan^{-1}\left(\frac{n}{k^2+k+n^2}\right)=tan^{-1}\left(\frac{k+1}{n}\right)-tan^{-1}\left(\frac{k}{n}\right)$$

Thus,

$$\lim_{n\to\infty}\sum_{k=0}^{n}tan^{-1}\left(\frac{n}{n^2+k^2+k}\right)=\lim_{n\to\infty}\sum_{k=0}^{n}tan^{-1}\left(\frac{n}{n^2+k(k+1)}\right)=$$

$$=\lim_{n\to\infty}\sum_{k=0}^{n}tan^{-1}\left(\frac{\frac{1}{n}}{\frac{k+1}{n}\cdot\frac{k}{n}+1}\right)=\lim_{n\to\infty}\sum_{k=0}^{n}tan^{-1}\left(\frac{\frac{k+1}{n}-\frac{k}{n}}{1+\frac{k+1}{n}\cdot\frac{k}{n}}\right)=$$

$$=\lim_{n\to\infty}\sum_{k=0}^{n}\left(tan^{-1}\left(\frac{k+1}{n}\right)-tan^{-1}\left(\frac{k}{n}\right)\right)=\lim_{n\to\infty}tan^{-1}\left(1+\frac{1}{n}\right)=\frac{\pi}{4}$$

Solution 3 by Mohammad Rostami-Kabul-Afghanistan

$$tan^{-1}\left(\frac{n}{n^2+k^2+k}\right)=\alpha+\beta\to tan(\alpha+\beta)=\frac{n}{n^2+k^2+k}\to$$

$$\frac{tan\alpha+tan\beta}{1-tan\alpha\cdot tan\beta}=\frac{n}{n^2+k^2+k}\to\frac{n^2(tan\alpha+tan\beta)}{n^2-n^2tan\alpha\cdot tan\beta}=\frac{n}{n^2+k^2+k}$$

$$\begin{cases}n^2(tan\alpha+tan\beta)=n\\-n^2tan\alpha\cdot tan\beta=k^2+k\end{cases}\to\begin{cases}S=tan\alpha+tan\beta=\frac{1}{n}\\P\backslash tan\alpha\cdot tan\beta=\frac{k^2+k}{n^2}\end{cases}$$

$$Z^2-SZ+P=0\to Z^2-\frac{1}{n}Z-\frac{k^2+k}{n^2}=0\to$$

$$Z=\frac{1}{2}\left(\frac{1}{n}\pm\sqrt{\left(\frac{2k+1}{n}\right)^2}\right)=\frac{1}{2}\left(\frac{1}{n}\pm\frac{2k+1}{n}\right)\to\begin{cases}Z=\frac{2k+2}{2n}=\frac{k+1}{n}=tan\alpha\\Z=-\frac{2k}{2n}=-\frac{k}{n}=tan\beta\end{cases}\to$$

$$\alpha=tan^{-1}\left(\frac{k+1}{n}\right),\beta=tan^{-1}\left(-\frac{k}{n}\right)$$

$$\lim_{n\to\infty}\sum_{k=0}^{n}tan^{-1}\left(\frac{n}{n^2+k^2+k}\right)=\lim_{n\to\infty}\sum_{k=1}^{n}(\alpha+\beta)=$$

$$=\lim_{n\to\infty}\sum_{k=0}^{n}\left(tan^{-1}\left(\frac{k+1}{n}\right)-tan^{-1}\left(\frac{k}{n}\right)\right)=\lim_{n\to\infty}tan^{-1}\left(1+\frac{1}{n}\right)=\frac{\pi}{4}$$

Note by editor:

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Many thanks to Florică Anastase-Romania for typed solutions.