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In ΔABC the following relationship holds:

$$a^2b+b^2c+c^2a\le 9R\sqrt{\frac{9R^4-48r^4}{2}}$$

Proposed by Marian Ursărescu-Romania
Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco, Solution 2 by Tran Hong-Dong Thap-Vietnam, Solution 3 by Eldeniz Hesenov-Georgia

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum a^2b=abc\sum\frac{a}{c}\overset{BCS}{\le}abc\sqrt{\left(\sum a^2\right)\left(\sum\frac{1}{c^2}\right)}\overset{Leibniz}{\le}abc\sqrt{9R^2\cdot\frac{1}{4r^2}}=$$

$$=6R^2s\overset{(1)}{\le}9R\sqrt{\frac{9R^4-48r^4}{2}}$$

$$(1)\Leftrightarrow 8R^2s^2\le 81R^4-432r^4$$

By Gerretsen inequality: $8R^2s^2\le 32r^4+32R^3r+24R^2r^2$

It is suffices to prove that: $49R^4\ge 32R^3r+24R^2r^2+432r^4$

By Euler inequality: $16R^4\ge 16\cdot 2r\cdot R^3=32R^3r$

$6R^4\ge 6\cdot(2r)^2R^2=24R^2r^2$ and $27R^4\ge 27\cdot(2r)^4=432r^4$, thus

$$49R^4 \ge 32R^3r + 24R^2r^2 + 432r^4$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

- $a^2b + b^2c + c^2a \le a^3 + b^3 + c^3 = 2s(s^2 - 6Rr - 3r^2)$

$$\overset{s\le\frac{4R+r}{\sqrt{3}}}{\le} \frac{2}{\sqrt{3}}(4R+r).(s^2-6Rr-3r^2) \overset{s^2\le 4R^2+4Rr+3r^2}{\le}$$

$$\frac{2}{\sqrt{3}}(4R+r)(4R^2-2Rr) \overset{(1)}{\le} 9R\sqrt{\frac{9R^4-48r^4}{2}};$$

$$(1) \leftrightarrow \frac{4}{3}(4R+r)^2.(4R-2r)^2 \le 81\left(\frac{9R^4-48r^4}{2}\right);$$

$$\leftrightarrow 3.81(9x^4-48) \ge 8.(4x+1)^2(4x-2)^2;\ \left(\because x = \frac{R}{r} \ge 2\right)$$

$$\leftrightarrow 139x^4 + 1024x^3 + 384x^2 - 128x - 11696 \ge 0;$$

$$\leftrightarrow (x-2)(139x^3 + 1302x^2 + 2988x + 5848) \ge 0;$$

$$(\because \text{true by } x \ge 2) \to (1) \text{ is true. Proved.}$$

Solution 3 by Eldeniz Hesenov-Georgia

$$LHS \overset{BCS}{\le} \sqrt{\left(\sum a^2b^2\right)\left(\sum a^2\right)} \overset{Leibniz}{\le} 3R\cdot 4Rrs\sqrt{\sum\frac{1}{a^2}} \le 6R^2s;\ (1)$$

$$(1) \Leftrightarrow 36s^2R^4 \le 81R^2\frac{9R^4-48r^4}{2}$$

$$4s^2 \le 27R^2 \Leftrightarrow 27R^4\cdot 2 \le 81R^4 - 9\cdot 48r^4 \Leftrightarrow$$

$$27R^4 \ge 9\cdot 48r^4 \Leftrightarrow R \ge 2r(Euler).$$

Note by editor:

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