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In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_b^2+m_c^2}{r_a}\le\sum_{cyc}\frac{m_b^2+m_c^2}{h_a}$$

Proposed by Marin Chirciu-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\sum_{cyc}\frac{m_b^2+m_c^2}{r_a}\le\sum_{cyc}\frac{m_b^2+m_c^2}{h_a}$$

$$\Leftrightarrow\left(\sum_{cyc}m_a^2\right)\left(\sum_{cyc}\frac{1}{r_a}\right)-\sum_{cyc}\frac{m_a^2}{r_a}\le\left(\sum_{cyc}m_a^2\right)\left(\sum_{cyc}\frac{1}{h_a}\right)-\sum_{cyc}\frac{m_a^2}{h_a}$$

$$\left(\sum_{cyc}\frac{1}{r_a}=\sum_{cyc}\frac{1}{h_a}=\frac{1}{r}\right)$$

$$\Leftrightarrow\sum_{cyc}\frac{m_a^2}{h_a}\le\sum_{cyc}\frac{m_a^2}{r_a}\Leftrightarrow\frac{1}{2}\sum_{cyc}am_a^2\le\sum_{cyc}(s-a)m_a^2\Leftrightarrow\frac{3}{2}\sum_{cyc}am_a^2\le s\sum_{cyc}m_a^2$$

$$\Leftrightarrow\frac{3}{2}\left[\frac{a(2(b^2+c^2)-a^2)}{4}+\frac{b(2(c^2+a^2)-b^2)}{4}+\frac{c(2(a^2+b^2)-c^2)}{4}\right]\le$$

$$\le\frac{a+b+c}{2}\cdot\frac{3}{4}(a^2+b^2+c^2)$$

$$\Leftrightarrow a(2b^2+2c^2-a^2)+b(2a^2+2c^2-b^2)+c(2a^2+2b^2-c^2)$$

$$\le (a+b+c)(a^2+b^2+c^2$$

$$\Leftrightarrow 2a^3+2b^3+2c^3 \ge (ab^2+bc^2+ca^2)+(ba^2+cb^2+ac^2);\ \ (1)$$

Which is clearly true because:

$$a^3+b^3+c^3 \ge ab^2+bc^2+ca^2;\ \ (2)$$

$$a^3+b^3+c^3 \ge ba^2+cb^2+ac^2;\ \ (3)$$

From (2),(3) it follows (1) is true. Proved.

Note by editor:

Many thanks to Florică Anastase-Romania for typed solution.