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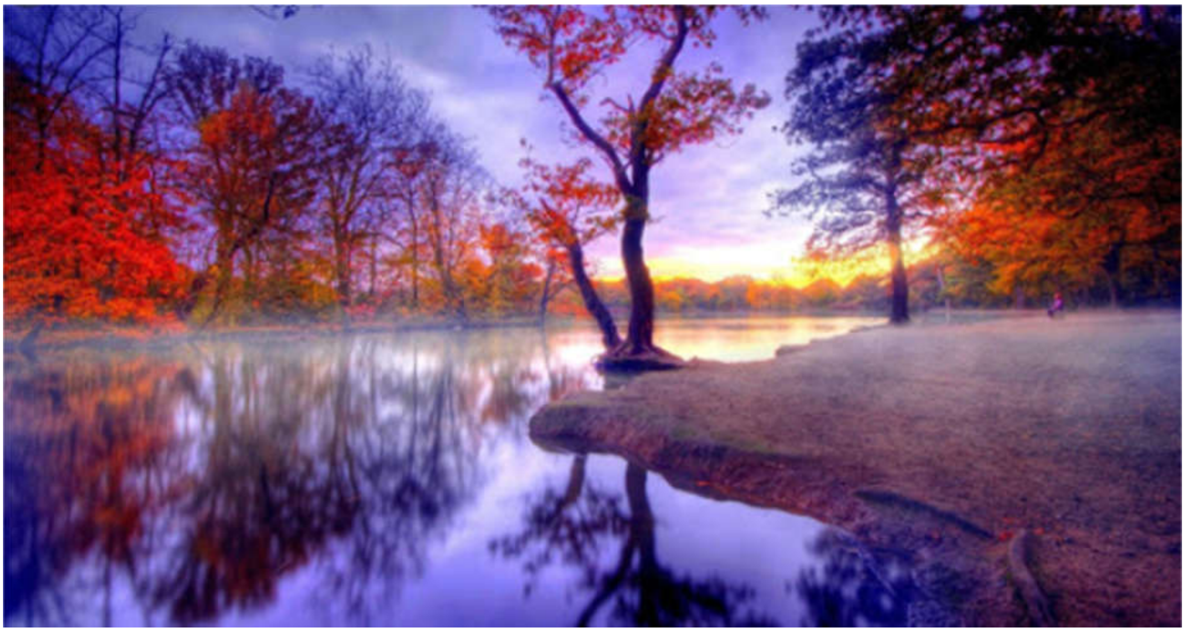

**If $a,b,c>0, abc=1, 0\le\lambda\le 2$ then:**

$$\frac{a^2}{b^2+\lambda c}+\frac{b^2}{c^2+\lambda a}+\frac{c^2}{a^2+\lambda b}\ge\frac{3}{\lambda+1}$$

***Proposed by Marin Chirciu-Romania***

***Solution by Tran Hong-Dong Thap-Vietnam***

- $\frac{a^2}{b^2+\lambda c}+\frac{b^2}{c^2+\lambda a}+\frac{c^2}{a^2+\lambda b}=\sum\frac{(a^2)^2}{a^2b^2+\lambda a^2c}\overset{C-S}{\ge}\frac{(\sum a^2)^2}{\sum a^2b^2+\lambda\sum ca^2}\overset{(1)}{\ge}\frac{3}{\lambda+1};$

  $(1)\leftrightarrow(\lambda+1)\left(\sum a^2\right)^2\ge 3\left(\sum a^2b^2+\lambda\sum ca^2\right);$

  $\leftrightarrow(\lambda+1)\left(\sum a^4+2\sum a^2b^2\right)\ge 3\left(\sum a^2b^2+\lambda\sum ca^2\right);$

  $\leftrightarrow(\lambda+1)\sum a^4\ge 3\lambda\sum ca^2+(1-2\lambda)\sum a^2b^2;$

  $\leftrightarrow\left(\sum a^4-\sum a^2b^2\right)+\lambda\left(\sum a^4+2\sum a^2b^2-3\sum ca^2\right)\ge 0;$

**Because:**

- $\sum x^2\ge\sum xy\to\sum a^4\ge\sum a^2b^2\to\sum a^4-\sum a^2b^2\ge 0;$ (2)
- $a^4+a^2c^2+a^2c^2\overset{AM-GM}{\ge}3.\sqrt[3]{a^6c^3a^2c}=3.a^2c.\sqrt[3]{a^2c};$
- **Similary:**

  $b^4+a^2b^2+a^2b^2\ge 3.b^2a.\sqrt[3]{b^2a};$

$c^4+b^2c^2+b^2c^2\geq 3.c^2b.\sqrt[3]{c^2b};$

$\rightarrow\sum a^4+2.\sum a^2b^2\geq 3.\sum\left(b^2a.\sqrt[3]{b^2a}\right)=3.\sum\varphi(b^2a)$

$$\overset{\text{JENSEN: }\varphi(x)=x\sqrt[3]{x}}{\geq}\quad 3.3.\varphi\left(\frac{b^2a+c^2b+a^2c}{3}\right)=3.3.\left(\frac{\sum b^2a}{3}\right).\sqrt[3]{\frac{\sum b^2a}{3}}$$

$$\overset{\text{AM-GM}}{\geq}\ 3.3.\left(\frac{\sum b^2a}{3}\right).\sqrt[3]{\frac{3.\sqrt[3]{(abc)^3}}{3}}=3.\sum b^2a.\sqrt[3]{\frac{3.1}{3}}=3.\sum b^2a;$$

$\overset{\lambda\geq 0}{\Rightarrow}\lambda\left(\sum a^4+2.\sum a^2b^2-3.\sum b^2a\right)\geq 0;$ (3)

$\overset{(2)+(3)}{\Rightarrow}$ (1) is true. Proved.