Interesting integral

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<u>`Question Proposed by Rana Ranino Setif,Algeria`</u>

`Prove that:`

$$\int_0^\infty x^2 e^{-x^2} \texttt{erf}(x)\log(x)dx = \left(\frac{2\pi - \gamma(\pi+2) - 2(1+\pi)\log(2) + 4G}{16\sqrt{\pi}}\right)$$

Solution

Let

$$\omega(n,x) = \int_0^\infty x^2 e^{-x^2} \texttt{erf}(nx)\log(x)dx \qquad \forall n > 0, \omega(0,x) = 0$$

for $x^2 = t$

$$\omega(n,t) = \frac{1}{2}\int_0^\infty \sqrt{t}e^{-t}\texttt{erf}(n\sqrt{t})\log(\sqrt{t})dt$$

$$\rightarrow \omega(n,t) = \frac{1}{4}\int_0^\infty \sqrt{t}e^{-t}\texttt{erf}(n\sqrt{t})\log(t)dt$$

$$\frac{\delta\omega(n,t)}{\delta n} = \frac{1}{2\sqrt{\pi}}\int_0^\infty te^{-t(1+n^2)}\log(t)dt$$

Taking

$$I = \int_0^\infty x^\alpha e^{-\beta x}dx$$

Putting $u = \beta x$

$$I = \int_0^\infty \frac{u^\alpha}{\beta^{\alpha+1}}e^{-u}du = \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}}$$

$$I(\alpha) = \frac{1}{2\sqrt{\pi}}\int_0^\infty t^\alpha e^{-t(1+n^2)}dt \qquad I'(1) = \frac{\delta\omega(n,t)}{\delta n}$$

$$I(\alpha) = \frac{1}{2\sqrt{\pi}}\left(\frac{\Gamma(\alpha+1)}{(1+n^2)^{\alpha+1}}\right)$$

$$I'(\alpha) = \frac{1}{2\sqrt{\pi}}\left(\frac{\Gamma(\alpha+1)(\psi(\alpha+1) - \log(1+n^2))}{(1+n^2)^{\alpha+1}}\right)$$

$$\frac{\delta\omega(n,t)}{\delta n} = \frac{1}{2\sqrt{\pi}}\left(\frac{\Gamma(2)(\psi(2)-\log(1+n^2))}{(1+n^2)^2}\right)$$

$$\frac{\delta\omega(n,t)}{\delta n} = \frac{1}{2\sqrt{\pi}}\left(\frac{(\psi(2)-\log(1+n^2))}{(1+n^2)^2}\right)$$

Integrating w.r.t n

$$\omega(n,t) = \frac{1}{2\sqrt{\pi}}\int_0^n \left(\frac{(\psi(2)-\log(1+x^2))}{(1+x^2)^2}\right)dx + C$$

$$\omega(0,t) = \frac{1}{2\sqrt{\pi}}\int_0^0 \left(\frac{(\psi(2)-\log(1+x^2))}{(1+x^2)^2}\right)dx + C$$

Recall $\int_a^b f(x)dx$ where a=b=0,Is zero.

Therefore C=0 We can define $\omega(n,t)$ as

$$\omega(n,t) = \frac{1}{2\sqrt{\pi}}\int_0^n \left(\frac{(\psi(2)-\log(1+x^2))}{(1+x^2)^2}\right)dx$$

For n=1

$$\omega(1,t) = \frac{1}{2\sqrt{\pi}}\int_0^1 \left(\frac{(\psi(2)-\log(1+x^2))}{(1+x^2)^2}\right)dx$$

This integral can be separated into two. Let

$$\omega_1(1,t) = \frac{\psi(2)}{2\sqrt{\pi}}\int_0^1 \frac{1}{(1+x^2)^2}dx$$

Putting $x = \tan u$

$$\omega_(1,t) = \frac{\psi(2)}{2\sqrt{\pi}}\int_0^{\frac{\pi}{4}} cos^2 u du = \frac{\psi(2)(\pi+2)}{16\sqrt{\pi}}$$

Also

$$\omega_2(1,t) = \frac{1}{2\sqrt{\pi}}\int_0^1 \left(\frac{\log(1+x^2)}{(1+x^2)^2}\right)dx$$

Putting $x = \tan t$

$$\omega_2(1,t) = \frac{1}{2\sqrt{\pi}}\int_0^{\frac{\pi}{4}} \left(\frac{\log(1+\tan^2 t)}{(1+\tan^2 t)^2}\right)\sec^2 t dx = \frac{1}{2\sqrt{\pi}}\int_0^{\frac{\pi}{4}} \frac{\log(\sec^2 t)}{\sec^2 t}dt$$

$$\omega_2(1,t) = -\frac{2}{2\sqrt{\pi}}\int_0^{\frac{\pi}{4}} \cos^2 t \log(\cos t)dt$$

Applying IBP

$$\omega_2(1,t) = \frac{-1}{\sqrt{\pi}}\left(\left(\frac{1}{2}\left(t+\sin t\cos t\right)\log(\cos(t))\right)_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}}\left(\frac{1}{2}\left(t+\sin t\cos t\right)\tan t dt\right)\right)$$

$$\omega_2(1,t) = \frac{-1}{\sqrt{\pi}}\left(-\frac{1}{16}(2+\pi)\log(2) + \frac{1}{2}\int_0^{\frac{\pi}{4}}\left(t\tan t + sin^2 t\right)dt\right)$$

$$\omega_2(1,t) = \frac{-1}{\sqrt{\pi}}\left(-\frac{1}{16}(2+\pi)\log(2) + \frac{\pi}{16} - \frac{1}{8} + \frac{1}{2}\int_0^{\frac{\pi}{4}} t\tan t dt\right)$$

Let

$$I = \int_0^{\frac{\pi}{4}} t\tan t dt$$

Applying IBP

$$I = \frac{\pi\log(2)}{8} + \int_0^{\frac{\pi}{4}} \log(\cos t)dt$$

Let

$$I_1 = \int_0^{\frac{\pi}{4}} \log(\sin t)dt \quad and \quad I_2 = \int_0^{\frac{\pi}{4}} \log(\cos t)dt$$

$$I_1 - I_2 = \int_0^{\frac{\pi}{4}} \log(\tan t)dt$$

$$I_1 - I_2 = \int_0^1 \frac{\log u}{1+u^2}du \quad u = \tan t$$

Since

$$G = -\int_0^1 \frac{\log u}{1+u^2}du, \quad I_1 - I_2 = -G$$

$$I_1 + I_2 = \int_0^{\frac{\pi}{4}} \log(\sin(2t)) - \log(2)dt$$

$$I_1 + I_2 = \frac{1}{2}\int_0^{\frac{\pi}{2}} \log(\sin(t))dt - \frac{\pi\log(2)}{4}$$

since $\int_0^{\frac{\pi}{2}} \log(\sin(t))dt$ is a well known integral

$$I_1 + I_2 = -\frac{\pi\log(2)}{4} + \frac{\pi\log(2)}{4} = -\frac{\pi\log(2)}{2}$$

Now

$$I_1 + I_2 - I_1 + I_2 = -\frac{\pi\log(2)}{2} + G \Rightarrow I_2 = \frac{G}{2} - \frac{\pi\log(2)}{4}$$

$$I = \frac{\pi\log(2)}{8} + \frac{G}{2} - \frac{\pi\log(2)}{4} = \frac{4G - \pi\log(2)}{8}$$

Since

$$\omega_2(1,t) = \frac{-1}{\sqrt{\pi}}\left(-\frac{1}{16}(2+\pi)\log(2) + \frac{\pi}{16} - \frac{1}{8} + \frac{1}{2}I\right)$$

$$\omega_2(1,t) = \frac{-1}{\sqrt{\pi}}\left(-\frac{1}{16}(2+\pi)\log(2) + \frac{\pi}{16} - \frac{2}{16} + \frac{4G - \pi\log(2)}{16}\right)$$

$$\omega_2(1,t) = \frac{-1}{\sqrt{\pi}}\left(\frac{-2\log(2) - 2\pi\log(2) + \pi - 2 + 4G}{16}\right)$$

Now

$$\omega(1,t) = \frac{\psi(2)(\pi+2)}{16\sqrt{\pi}} + \left(\frac{-2\log(2) - 2\pi\log(2) + \pi - 2 + 4G}{16\sqrt{\pi}}\right)$$

$$\omega(1,t) = \frac{(\pi+2)\psi(2) - 2(1+\pi)\log(2) + \pi - 2 + 4G}{16\sqrt{\pi}}$$

Since $\psi(2) = 1 - \gamma$

$$\omega(1,t) = \frac{(\pi+2)(1-\gamma) - 2(1+\pi)\log(2) + \pi - 2 + 4G}{16\sqrt{\pi}}$$

$$\omega(1,t) = \frac{2\pi - \gamma(\pi+2) - 2(1+\pi)\log(2) + 4G}{16\sqrt{\pi}}$$

Hence

$$\int_0^\infty x^2 e^{-x^2}\texttt{erf}(x)\log(x)dx = \left(\frac{2\pi - \gamma(\pi+2) - 2(1+\pi)\log(2) + 4G}{16\sqrt{\pi}}\right)$$