ABOUT TRIANGLE UVW OF MEHMET ŞAHIN

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Let be ΔABC with the area F, the semiperimeter s and the other usual notations.

Theorem 1. If $x, y, z > 0$, then Tsintsifas inequality holds:

$$\text{(T2)} \qquad \frac{x}{y+z}\cdot a^4 + \frac{y}{z+x}\cdot b^4 + \frac{z}{x+y}\cdot c^4 \geq 8F^2$$

Proof. We have:

$$\sum_{cyc}\frac{x}{y+z}\cdot a^4 = \sum_{cyc}\frac{(xa^2)^2}{xy+xz} \overset{\text{Bergström}}{\geq} \frac{(xa^2+yb^2+zc^2)^2}{\sum_{cyc}(xy+xz)} =$$

$$= \frac{(xa^2+yb^2+zc^2)^2}{2(xy+yz+zx)} \overset{\text{Klamkin}}{\geq} \frac{16(xy+yz+xz)S^2}{2(xy+yz+zx)} = 8F^2$$

□

In problem 11857 from A.M.M, 2015, pp. 700, Mehmet Şahin proves that:
If UVW is a triangle with the sides $u = \sqrt{a}, v = \sqrt{b}, w = \sqrt{c}$ triangle ABC then denoting $\Delta =$ area UVW we have:

$$\text{(M.S)} \qquad \Delta = \frac{1}{2}\sqrt{r(r_a+r_b+r_c)}$$

Indeed, we have:

$$\text{(1)} \qquad r = \frac{F}{s} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

and

$$\text{(2)} \qquad r_a = \frac{F}{s-a} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-a} = \sqrt{\frac{s(s-b)(s-c)}{s-a}}$$

and the analogs. So,

$$r\cdot r_a = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}\sqrt{\frac{s(s-b)(s-c)}{s-a}} = (s-b)(s-c)$$

and the analogs according to Heron's formula, we have:

$$\Delta = \sqrt{\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{2}\cdot\frac{-\sqrt{a}+\sqrt{b}+\sqrt{c}}{2}\cdot\frac{\sqrt{a}-\sqrt{b}+\sqrt{c}}{2}\cdot\frac{\sqrt{a}+\sqrt{b}-\sqrt{c}}{2}} =$$

$$= \frac{1}{4}\sqrt{(\sqrt{a}+\sqrt{b}+\sqrt{c})(-\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}-\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c})} =$$

$$= \frac{1}{4}\sqrt{((\sqrt{b}+\sqrt{c})^2-a)(a-(\sqrt{b}-\sqrt{c})^2)} = \frac{1}{4}\sqrt{(2\sqrt{bc}+(b+c-a))(2\sqrt{bc}-(b+c-a))} =$$

$$= \frac{1}{4}\sqrt{4bc-(b+c-a)^2} = \frac{1}{4}\sqrt{4bc-a^2-b^2+c^2+2ab+2ac-2bc}$$

$$=\frac{1}{4}\sqrt{2ab+2bc+2ca-a^2-b^2-c^2}=\frac{1}{4}\sqrt{a^2-(b-c)^2+b^2-(c-a)^2+c^2-(a-b)^2}=$$

$$=\frac{1}{4}\sqrt{4((s-b)(s-c)+(s-c)(s-a)+(s-a)(s-b))}=$$

$$=\frac{1}{2}\sqrt{r\cdot r_a+r\cdot r_b+r\cdot r_c}=\frac{1}{2}\sqrt{r(r_a+r_b+r_c)}=\frac{1}{2}\sqrt{r(4R+r)} \tag{3}$$

Theorem 2. If $x,y,z>0$, then the following inequality holds:

$$\frac{x}{y+z}\cdot a^2+\frac{y}{z+x}\cdot b^2+\frac{z}{x+y}\cdot c^2\geq 2\sqrt{3}F \tag{T1}$$

Proof. In triangle UVW we apply inequality (T2) and we obtain:

$$\sum_{cyc}\frac{xa^2}{y+z}=\sum_{cyc}\frac{x}{y+z}u^4\geq 8\Delta^2\Leftrightarrow\sum_{cyc}\frac{x}{y+z}(\sqrt{a})^4\geq 8\cdot\frac{1}{4}r(4R+r)=$$

$$=2\cdot r(4R+r)\overset{\text{Doucet}}{\geq}2\cdot r\cdot s\cdot\sqrt{3}=2\sqrt{3}s\cdot r=2\sqrt{3}F$$

□

Theorem 3. If $x,y,z>0$ then the following inequality holds:

$$\frac{x}{y+z}\cdot a+\frac{y}{z+x}\cdot b+\frac{z}{x+y}\cdot c=\sqrt[4]{27}\cdot\sqrt{F} \tag{4}$$

Proof. In triangle UVW we apply inequaity (T1) and we obtain:

$$\sum_{cyc}\frac{xa}{y+z}=\sum_{cyc}\frac{x}{y+z}u^2\geq 2\sqrt{3}\Delta\overset{(3)}{=}2\sqrt{3}\cdot\frac{1}{2}\sqrt{r(4R+r)}=$$

$$=\sqrt{3}\cdot\sqrt{r(4R+r)}\overset{\text{Doucet}}{\geq}\sqrt{3}\sqrt{r(s\sqrt{3})}=\sqrt{3}\cdot\sqrt[4]{3}\cdot\sqrt{sr}=\sqrt[4]{27}\sqrt{F}$$

□

Theorem 4. If $x,y,z>0$ then in any ΔABC the following inequality holds:

$$xa+yb+zc\geq 2\sqrt[4]{3}\sqrt{xy+yz+zx}\cdot\sqrt{F} \tag{5}$$

Proof. We consider the associate triangle UVW so we apply:

$$xa+yb+zc=xu^2+yv^2+zw^2\geq 4\sqrt{xy+yz+zx}\cdot\Delta=$$

$$=4\cdot\sqrt{xy+yz+zx}\cdot\frac{1}{2}\sqrt{r(r_a+r_b+r_c)}=2\sqrt{xy+yz+zx}\cdot\sqrt{r(4R+r)}\geq$$

$$\overset{\text{Doucet}}{\geq}2\sqrt{xy+yz+zx}\cdot\sqrt{r\cdot s\sqrt{3}}=2\cdot\sqrt[4]{3}\cdot\sqrt{xy+yz+zx}\cdot\sqrt{F}$$

□

Theorem 5. If $m\geq 0, x,y,z>0$ then in any ΔABC the following inequality holds:

$$x\cdot a^{m+1}+y\cdot b^{m+1}+z\cdot c^{m+1}\geq\frac{(2\sqrt[4]{3}\cdot\sqrt{xy+yz+zx}\cdot\sqrt{F})^{m+1}}{(x+y+z)^m} \tag{6}$$

Proof. We have:

$$\sum_{cyc} x\cdot a^{m+1} = \sum_{cyc}\frac{(xa)^{m+1}}{x^m} \overset{\text{J. Radon}}{\geq} \frac{\left(\sum_{cyc} xa\right)^{m+1}}{(x+y+z)^m} \overset{(5)}{\geq}$$

$$\geq \frac{2^{m+1}\cdot 3^{\frac{m+1}{4}}\cdot (xy+yz+zx)^{\frac{m+1}{2}}\cdot F^{\frac{m+1}{2}}}{(x+y+z)^m} = \frac{(2\sqrt[4]{3}\cdot\sqrt{xy+yz+zx}\cdot\sqrt{F})^{m+1}}{(x+y+z)^m}$$

□

Theorem 6. If $x,y,z>0$, then in any ΔABC the following inequality holds:

$$\frac{xa^2}{(y+z)^2}+\frac{yb^2}{(z+x)^2}+\frac{zc^2}{(x+y)^2} \geq \frac{3\sqrt{3}}{x+y+z}F \tag{7}$$

Proof. We consider the associate triangle UVW with the sides $u=\sqrt{a}, v=\sqrt{b}$, $w=\sqrt{c}$ and the area Δ, then:

$$\sum_{cyc}\frac{xa^2}{(y+z)^2} = \sum_{cyc}\frac{xu^4}{(y+z)^2} = \sum_{cyc}\frac{\left(\frac{xu^2}{y+z}\right)^2}{x} \overset{\text{Bergström}}{\geq} \frac{\left(\sum_{cyc}\frac{xu^2}{y+z}\right)^2}{x+y+z} \geq$$

$$\overset{(T1)}{\geq} \frac{(2\sqrt{3}\Delta)^2}{x+y+z} = \frac{12\Delta^2}{x+y+z} = \frac{12}{x+y+z}\left(\frac{1}{2}\sqrt{r(4R+r)}\right)^2 =$$

$$= \frac{12}{4(x+y+z)}\cdot r(4R+r) \overset{\text{Doucet}}{\geq} \frac{3}{x+y+z}\cdot r\cdot s\cdot\sqrt{3} = \frac{3\sqrt{3}F}{x+y+z}$$

□

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