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ROMANIAN MATHEMATICAL MAGAZINE

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ABOUT AN INEQUALITY BY NGUYEN VAN CANH-VIII

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1) In ΔABC the following relationship holds:

$$\boldsymbol{w_a^2+w_b^2+w_c^2+8r(R-2r)\ge s^2}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution. **Lemma. 2) In ΔABC the following relationship holds:**

$$\sum_{cyc} w_a^2=\frac{s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+r^2+2Rr)^2}$$

Proof. Using $w_a=\frac{2bc}{b+c}cos\frac{A}{2}$, we get:

$$\sum_{cyc} w_a^2=\sum_{cyc}\left(\frac{2bc}{b+c}cos\frac{A}{2}\right)^2=4\sum_{cyc}\frac{b^2c^2}{(b+c)^2}\cdot\frac{s(s-a)}{bc}=4s\sum_{cyc}\frac{bc(s-a)}{(b+c)^2}=$$

$$=4s\cdot\frac{s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{4s(s^2+r^2+2Rr)^2}=$$

$$=\frac{s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+r^2+2Rr)^2}$$

$$\because \sum_{cyc}\frac{bc(s-a)}{(b+c)^2}=\frac{s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{4s(s^2+r^2+2Rr)^2}$$

Let's get back to the main problem. Using Lemma, inequality can be written as:

$$\frac{s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{4s(s^2+r^2+2Rr)^2}+8r(R-2r)\ge s^2 \Leftrightarrow$$

$$s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2+$$
$$+8r(R-2r)(s^2+r^2+2Rr)^2\ge s^2(s^2+r^2+2Rr)^2 \Leftrightarrow$$

$$s^4(4Rr-15r^2)+s^2r^2(60R^2-12Rr-30r^2)+r^3(32R^3-16R^2r-48Rr^2-15r^3)\ge 0$$

$$s^4(4R-15r)+s^2r(60R^2-12Rr-30r^2)+r^2(32R^3-16R^2r-48Rr^2-15r^3)\ge 0$$

Distinguish the cases:

Case I) If $(4R-15r)\ge 0$ inequality is obviously true.

Case II) If $(4R-15r)<0$, inequality can be written as:

$$s^2r(60R^2-12Rr-30r^2)+r^2(32R^3-16R^2r-48Rr^2-15r^3)\ge s^4(15r-4R)$$

which follows from $16Rr-5r^2\le s^2\le 4R^2+4Rr+3r^2(Gerretsen)$.

$$r(16Rr-5r^2)(60R^2-12Rr-30r^2)+r^2(32R^3-16R^2r-48Rr^2-15r^3)$$
$$\ge(4R^2+4Rr+3r^2)^2(15r-4R)$$

$$r^2(16R-5r)(60R^2-12Rr-30r^2)+r^2(32R^3-16R^2r-48Rr^2-15r^3)$$
$$\ge(4R^2+4Rr+3r^2)^2(15r-4R)$$

$$r^2(960R^3-492R^2r-420Rr^2+150r^3)+r^2(32R^3-16R^2r-48Rr^2-15r^3)$$
$$\ge(16R^4+32R^3r+40R^2r^2+24Rr^3+9r^4)$$
$$r^2(992R^3-508R^2r-468Rr^2+135r^3)$$
$$\ge(16R^4+32R^3r+40R^2r^2+24Rr^3+9r^4)(15r-4R)$$
$$64R^5-112R^4r+672R^3r^2-1012R^2r^3-792Rr^4\ge 0$$
$$16R^4-28R^3r+168R^2r^2-253Rr^3-198r^4\ge 0$$
$(R-2r)(16R^3+4R^2r+176Rr^2+99r^3)\ge 0$, which is true from $R\ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

3) In ΔABC the following relationship holds:

$$w_a^2+w_b^2+w_c^2+nr(R-2r)\ge s^2, n\ge 4$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma. 4) In ΔABC the following relationship holds:**

$$\sum_{cyc} w_a^2=\frac{s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+r^2+2Rr)^2}$$

Proof. Using $w_a=\frac{2bc}{b+c}cos\frac{A}{2}$, we get:

$$\sum_{cyc} w_a^2=\sum_{cyc}\left(\frac{2bc}{b+c}cos\frac{A}{2}\right)^2=4\sum_{cyc}\frac{b^2c^2}{(b+c)^2}\cdot\frac{s(s-a)}{bc}=4s\sum_{cyc}\frac{bc(s-a)}{(b+c)^2}=$$
$$=4s\cdot\frac{s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{4s(s^2+r^2+2Rr)^2}=$$
$$=\frac{s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+r^2+2Rr)^2}$$
$$\because\sum_{cyc}\frac{bc(s-a)}{(b+c)^2}=\frac{s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{4s(s^2+r^2+2Rr)^2}$$

Let's get back to the main problem. Using Lemma, inequality can be written as:

$$\frac{s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+r^2+2Rr)^2}+nr(R-2r)\ge s^2$$
$$s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2+nr(R-2r)(s^2+r^2+2Rr)^2$$
$$\ge s^2(s^2+r^2+2Rr)^2$$
$$s^4[(n-4)Rr+(1-2n)r^2]+s^2r^2[(4n+28)R^2+(36-6n)Rr+(2-4n)r^2]$$
$$+r^3[4nR^3+(16-4n)R^2r+(8-7n)Rr^2+(1-2n)r^3]\ge 0$$
$$s^4[(n-4)R+(1-2n)r]+s^2r[(4n+28)R^2+(36-6n)Rr+(2-4n)r^2]$$
$$+r^2[4nR^3+(16-4n)R^2r+(8-7n)Rr^2+(1-2n)r^3]\ge 0$$

Distinguish the cases:

Case I) If $[(n-4)R+(1-2n)r]\ge 0$, inequality is obviously true.

Case II) If $[(n-4)R+(1-2n)r]<0$, inequality can be written as:

$$s^2r[(4n+28)R^2+(36-6n)Rr+(2-4n)r^2]+$$

$$+r^2[4nR^3+(16-4n)R^2r+(8-7n)Rr^2+(1-2n)r^3]\geq s^4[(4-n)R+(2n-1)r]$$

Which follows from $16Rr-5r^2\leq s^2\leq 4R^2+4Rr+3r^2(Gerretsen)$. Remains to prove that:

$$r(16Rr-5r^2)[(4n+28)R^2+(36-6n)Rr+(2-4n)r^2]+$$
$$+r^2[4nR^3+(16-4n)R^2r+(8-7n)Rr^2+(1-2n)r^3]$$
$$\geq(4R^2+4Rr+3r^2)[(4-n)R+(2n-1)r]\Leftrightarrow$$
$$r^2[(64n+448)R^3+(-116n+436)R^2r+(-34n-148)Rr^2+(20n-10)r^3]+$$
$$+r^2[4nR^3+(16-4n)R^2r+(8-7n)Rr^2+(1-2n)r^3]\geq$$
$$\geq(16R^4+32R^3r+40R^2r^2+24Rr^3+9r^4)[(4-n)R+(2n-1)r]\Leftrightarrow$$
$$r^2[(68n+448)R^3+(-120n+452)R^2r+(-41n-140)Rr^2+(18n-9)r^3]\geq$$
$$\geq 16(4-n)R^5+112R^4r+(24n+128)R^3r^2+(56n+56)R^2r^3+(39n+12)Rr^4$$
$$+9(2n-1)r^5\Leftrightarrow$$
$$16(4-n)R^5-112R^4r+(44n+320)R^3r^2+(-176n+396)R^2r^3+(-80n-152)Rr^4$$
$$\geq 0\Leftrightarrow$$
$$4(n-4)R^4-28R^3r+(11n+80)R^2r^2+(-44n+99)Rr^3-(20n+38)r^4\geq 0\Leftrightarrow$$

$(R-2r)[4(n-4)R^3+(8n-60)R^2r+(27n-40)Rr^2+(10n+19)r^3]\geq 0$, which is true from $R\geq 2r(Euler)$. Equality holds if and only if triangle is equilateral.

Note.

1) **For $n=4$, we get $w_a^2+w_b^2+w_c^2+4r(R-2r)\geq s^2$**
2) **For $n=8$, we get Inequality in triangle -1811 from RMM 8/2020, proposed by Nguyen Van Canh-Vietnam.**

5) In ΔABC the following relationship holds:

$$w_a^2+w_b^2+w_c^2+4OI^2\geq s^2$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma. 6) In ΔABC the following relationship holds:**

$$\sum_{cyc}w_a^2=\frac{s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+r^2+2Rr)^2}$$

Proof. Using $w_a=\frac{2bc}{b+c}\cos\frac{A}{2}$, we get:

$$\sum_{cyc}w_a^2=\sum_{cyc}\left(\frac{2bc}{b+c}\cos\frac{A}{2}\right)^2=4\sum_{cyc}\frac{b^2c^2}{(b+c)^2}\cdot\frac{s(s-a)}{bc}=4s\sum_{cyc}\frac{bc(s-a)}{(b+c)^2}=$$
$$=4s\cdot\frac{s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{4s(s^2+r^2+2Rr)^2}=$$
$$=\frac{s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+r^2+2Rr)^2}$$
$$\because\sum_{cyc}\frac{bc(s-a)}{(b+c)^2}=\frac{s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{4s(s^2+r^2+2Rr)^2}$$

Let's get back to the main problem. Using Lemma and $OI^2 = R(R-2r)$, inequality can be written as:

$$\frac{s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+r^2+2Rr)^2}+4R(R-2r)\ge s^2 \Leftrightarrow$$

$$s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2+4R(R-2r)(s^2+r^2+2Rr)^2 \ge s^2(s^2+r^2+2Rr)^2$$

$$s^4(4R^2-12Rr+r^2)+s^2r(16R^3+4R^2r+20Rr^2+2r^3)+r^2(16R^4-16R^3r-12R^2r^2+r^4)\ge 0$$

Distinguish the cases:

Case I) If $(4R^2-12Rr+r^2)\ge 0$ inequality is obviously true.

Case II) If $(4R^2-12Rr+r^2)<0$, inequality can be written as:

$s^2r(16R^3+4R^2r+20Rr^2+2r^3)+r^2(16R^4-16R^3r-12R^2r^2+r^4)\ge$
$\ge s^4(-4R^2+12Rr-r^2)$, which follows from

$16Rr-5r^2\le s^2\le 4R^2+4Rr+3r^2 (Gerretsen)$. Remains to prove that:

$$(16Rr-5r^2)r(16R^3+4R^2r+20Rr^2+2r^3)+r^2(16R^4-16R^3r-12R^2r^2+r^4)\ge (4R^2+4Rr+3r^2)^2(-4R^2+12Rr-r^2)\Leftrightarrow$$

$$r^2(256R^4-16R^3r+300R^2r^2-68Rr^3-10r^4)+r^2(16R^4-16R^3r-12R^2r^2+r^4)\ge(16R^4+32R^3r+40R^2r^2+24Rr^3+9r^4)(-4R^2+12Rr-r^2)\Leftrightarrow$$

$$r^2(272R^4-32R^3r+288R^2r^2-80Rr^3-9r^4)\ge(16R^4+32R^3r+40R^2r^2+24Rr^3+9r^4)(-4R^2+12Rr-r^2)\Leftrightarrow$$

$$64R^6-64R^5r+64R^4r^2-384R^3r^3+76R^2r^4-164Rr^5\ge 0\Leftrightarrow$$

$$16R^6-16R^5r+16R^4r^2-96R^3r^3+19R^2r^4-41Rr^5\ge 0\Leftrightarrow$$

$(R-2r)(16R^3+4R^2r+176Rr^2+99r^3)\ge 0$, which is true from $R\ge 2r (Euler)$.

Equality holds if and only if triangle is equilateral.

7) In ΔABC the following relationship holds:

$$w_a^2+w_b^2+w_c^2+nOI^2\ge s^2$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma. 8) In ΔABC the following relationship holds:**

$$\sum_{cyc} w_a^2=\frac{s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+r^2+2Rr)^2}$$

Proof. Using $w_a=\frac{2bc}{b+c}cos\frac{A}{2}$, we get:

$$\sum_{cyc} w_a^2=\sum_{cyc}\left(\frac{2bc}{b+c}cos\frac{A}{2}\right)^2=4\sum_{cyc}\frac{b^2c^2}{(b+c)^2}\cdot\frac{s(s-a)}{bc}=4s\sum_{cyc}\frac{bc(s-a)}{(b+c)^2}=$$

$$=4s\cdot\frac{s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{4s(s^2+r^2+2Rr)^2}=$$

$$=\frac{s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+r^2+2Rr)^2}$$

$$\because\sum_{cyc}\frac{bc(s-a)}{(b+c)^2}=\frac{s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{4s(s^2+r^2+2Rr)^2}$$

Let's get back to the main problem. Using Lemma and $OI^2=R(R-2r)$, inequality can be written as:

$$\frac{s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+r^2+2Rr)^2}+nR(R-2r)\geq s^2$$

$$s^6+3s^4r^2+s^2r^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2+nR(R-2r)(s^2+r^2+2Rr)^2$$
$$\geq s^2(s^2+r^2+2Rr)^2\Leftrightarrow$$

$$s^4[nR^2-(2n+4)Rr+r^2]+s^2r[4nR^3+(28-6n)R^2r+(36-4n)Rr^2+2r^3]\geq$$
$$\geq r^2[4nR^4-4nR^3r+(16-7n)R^2r^2+(8-4n)Rr^3+r^4]\geq 0$$

Distinguish the cases:

Case I) If $[nR^2-(2n+4)Rr+r^2]\geq 0$, inequality is obviously true.

Case II) If $[nR^2-(2n+4)Rr+r^2]<0$, inequality can be written as:

$$s^2r[4nR^3+(28-6n)R^2r+(36-4n)Rr^2+2r^3]+$$
$$+r^2[4nR^4-4nR^3r+(16-7n)R^2r^2+(8-4n)Rr^3+r^4]\geq$$
$$\geq s^4[-nR^2+(2n+4)Rr-r^2],$$ which follows from

$16Rr-5r^2\leq s^2\leq 4R^2+4Rr+3r^2(Gerretsen)$. Remains to prove that:

$$(16Rr-5r^2)r[4nR^3+(28-6n)R^2r+(36-4n)Rr^2+2r^3]+$$
$$+r^2[4nR^4-4nR^3r+(16-7n)R^2r^2+(8-4n)Rr^3+r^4]\geq$$
$$\geq(4R^2+4Rr+3r^2)^2[-nR^2+(2n+4)Rr-r^2]\Leftrightarrow$$

$$r^2[64nR^4+(448-116n)R^3r+(436-34n)R^2r^2+(20n-148)Rr^3-10r^4]+$$
$$+r^2[4nR^4-4nR^3r+(16-7n)R^2r^2+(8-4n)Rr^3+r^4]\geq$$
$$\geq(16R^4+32R^3r+40R^2r^2+24Rr^3+9R^4)[-nR^4+(2n+4)Rr-r^2]\Leftrightarrow$$

$$r^2[648R^4+(448-120n)R^3r+(452-41n)R^2r^2+(18n-140)Rr^3-9r^4]\geq$$
$$\geq -16nR^6+64R^5r+(24n+112)R^4r^2+(56n+128)R^3r^3+(37n+56)R^2r^4$$
$$+(18n+12)Rr^5-9r^6\Leftrightarrow$$

$$16nR^6-64R^5r+(536-24n)R^4r^2+(320-176n)R^3r^3+(396-78n)R^2r^4$$
$$-152Rr^5\geq 0\Leftrightarrow$$

$$8nR^5-32R^4r+(268-12n)R^3r^2+(160-44n)R^2r^3+9198-39n)Rr^4-76r^5\geq 0$$
$\Leftrightarrow(R-2r)(16R^3+4R^2r+176Rr^2+99r^3)\geq 0$, which is true from $R\geq 2r(Euler)$.

Equality if and only if triangle is equilateral.

REFERENCE:

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