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ABOUT AN INEQUALITY BY NGUYEN VIET HUNG-I

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1) If $a,b,c>0$ then:

$$\frac{ab}{(a+b)^2}+\frac{bc}{(b+c)^2}+\frac{ca}{(c+a)^2}+\frac{(a+b)(b+c)(c+a)}{16abc}\ge\frac{5}{4}$$

Proposed by Nguyen Viet Hung-Vietnam

Solution. Using AM-GM inequality, we get:

$$LHS=\sum_{cyc}\frac{bc}{(b+c)^2}+\frac{1}{16}\prod_{cyc}\frac{b+c}{a}=\sum_{cyc}\frac{bc}{(b+c)^2}+\frac{1}{32}\prod_{cyc}\frac{b+c}{a}+\frac{1}{32}\prod_{cyc}\frac{b+c}{a}\ge$$

$$\ge5\sqrt[5]{\prod_{cyc}\frac{ab}{(a+b)^2}\cdot\frac{1}{32}\prod_{cyc}\frac{b+c}{a}\cdot\frac{1}{32}\prod_{cyc}\frac{b+c}{a}}=\frac{5}{4}=RHS.$$

Equality holds if and only if $a=b=c$.

Remark. The problem can be developed.

2) If $a,b,c>0$ then:

$$\frac{\sqrt{ab}}{a+b}+\frac{\sqrt{bc}}{b+c}+\frac{\sqrt{ca}}{c+a}+\frac{(a+b)(b+c)(c+a)}{16abc}\ge2$$

Proposed by Marin Chirciu-Romania

Solution. Using AM-GM inequality, we get:

$$RHS=\frac{\sqrt{ab}}{a+b}+\frac{\sqrt{bc}}{b+c}+\frac{\sqrt{ca}}{c+a}+\frac{(a+b)(b+c)(c+a)}{16abc}=\sum_{cyc}\frac{\sqrt{bc}}{b+c}+\frac{1}{16}\prod_{cyc}\frac{b+c}{a}\ge$$

$$\ge4\sqrt[4]{\prod_{cyc}\frac{\sqrt{bc}}{b+c}\cdot\frac{1}{16}\prod_{cyc}\frac{b+c}{a}}=4\sqrt[4]{\frac{1}{16}}=2=RHS.$$

Equality holds if and only if $a=b=c$.

3) If $a,b,c>0$ then:

$$\frac{ab\sqrt{ab}}{(a+b)^3}+\frac{bc\sqrt{bc}}{(b+c)^3}+\frac{ca\sqrt{ca}}{(c+a)^3}+\frac{3(a+b)(b+c)(c+a)}{64abc}\geq\frac{3}{4}$$

Proposed by Marin Chirciu-Romania

Solution. Using AM-GM inequality, we get:

$$LHS=\sum_{cyc}\frac{bc\sqrt{bc}}{(b+c)^3}+\frac{3}{64}\prod_{cyc}\frac{b+c}{a}=$$

$$=\sum_{cyc}\frac{bc\sqrt{bc}}{(b+c)^3}+\frac{1}{64}\prod_{cyc}\frac{b+c}{a}+\frac{1}{64}\prod_{cyc}\frac{b+c}{a}+\frac{1}{64}\prod_{cyc}\frac{b+c}{a}\geq$$

$$\geq 6\sqrt[6]{\prod_{cyc}\frac{bc\sqrt{bc}}{(b+c)^3}\cdot\frac{1}{64}\prod_{cyc}\frac{b+c}{a}\cdot\frac{1}{64}\prod_{cyc}\frac{b+c}{a}\cdot\frac{1}{64}\prod_{cyc}\frac{b+c}{a}}=\frac{3}{4}=RHS.$$

Equality holds if and only if $a=b=c$.

4) If $a,b,c>0$ then:

$$\frac{a^2b^2}{(a+b)^4}+\frac{b^2c^2}{(b+c)^4}+\frac{c^2a^2}{(c+a)^4}+\frac{(a+b)(b+c)(c+a)}{32abc}\geq\frac{7}{16}$$

Proposed by Marin Chirciu-Romania

Solution. Using AM-GM inequality, we get:

$$LHS=\sum_{cyc}\frac{b^2c^2}{(b+c)^4}+\frac{1}{32}\prod_{cyc}\frac{b+c}{a}=\sum_{cyc}\frac{b^2c^2}{(b+c)^4}+\frac{1}{128}\prod_{cyc}\frac{b+c}{a}+$$

$$+\frac{1}{128}\prod_{cyc}\frac{b+c}{a}+\frac{1}{128}\prod_{cyc}\frac{b+c}{a}+\frac{1}{128}\prod_{cyc}\frac{b+c}{a}\geq$$

$$\geq 7\sqrt[7]{\prod_{cyc}\frac{b^2c^2}{(b+c)^4}\cdot\frac{1}{128}\prod_{cyc}\frac{b+c}{c}\cdot\frac{1}{128}\prod_{cyc}\frac{b+c}{c}\cdot\frac{1}{128}\prod_{cyc}\frac{b+c}{c}\cdot\frac{1}{128}\prod_{cyc}\frac{b+c}{c}}=\frac{7}{16}=$$

RHS.Equality holds if and only if $a=b=c$.

5) If $a,b,c>0$ and $2n\in N$ then:

$$\frac{a^nb^n}{(a+b)^{2n}}+\frac{b^nc^n}{(b+c)^{2n}}+\frac{c^na^n}{(c+a)^{2n}}+\frac{2n(a+b)(b+c)(c+a)}{2^{2n+3}abc}\geq\frac{2n+3}{2^{2n}}$$

Proposed by Marin Chirciu-Romania

Solution. For $n=0$, we obtain equality $3=3$. Using AM-GM inequality, we get:

$$LHS=\sum_{cyc}\frac{b^nc^n}{(b+c)^{2n}}+\frac{2n}{2^{2n+3}}\prod_{cyc}\frac{b+c}{a}=$$

$$=\sum_{cyc}\frac{b^nc^n}{(b+c)^{2n}}+\frac{1}{2^{2n+3}}\prod_{cyc}\frac{b+c}{a}+\frac{1}{2^{2n+3}}\prod_{cyc}\frac{b+c}{a}+\cdots+\frac{1}{2^{2n+3}}\prod_{cyc}\frac{b+c}{a}\geq$$

$$\geq(2n+3)\sqrt[2n+3]{\prod_{cyc}\frac{b^nc^n}{(b+c)^{2n}}\cdot\frac{1}{2^{2n+3}}\prod_{cyc}\frac{b+c}{a}\cdot\frac{1}{2^{2n+3}}\prod_{cyc}\frac{b+c}{a}\cdot\ldots\cdot\frac{1}{2^{2n+3}}\prod_{cyc}\frac{b+c}{a}}=$$

$$=(2n+3)\sqrt[2n+3]{\left(\frac{1}{2^{2n+3}}\right)^{2n}}=\frac{2n+3}{2^{2n}}=RHS.$$

Equality holds if and only if $a=b=c$.

Note.

For $n=1$ we obtain Proposed Problem by Nguyen Viet Hung-Vietnam.

REFERENCES:

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