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ABOUT AN INEQUALITY BY HOANG LE NHAT TUNG-IV

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Note.

For $n = 2$ we obtain Problem SP.333-RMM, proposed by Hoang Le Nhat Tung-Hanoi-Vietnam.

If $x, y, z > 0, x + y + z = 3$ and $n \in \mathbb{N}$ find the minimum of expression:

$$P = \sum_{cyc} \frac{x}{n\sqrt{y} + \sqrt{z}} + \frac{3}{16(n+1)} \prod_{cyc} (y+z)$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma 1. If $x, y, z > 0, x + y + z = 3$ and $n \in N$ then:**

$$\sum_{cyc} \frac{x}{n\sqrt{y} + \sqrt{z}} \geq \frac{3\sqrt{3}}{(n+1)\sqrt{xy + yz + zx}}$$

Proof. Using BCS inequality, we have:

$$n\sqrt{y} + \sqrt{z} = \sqrt{y} + \sqrt{y} + \cdots + \sqrt{y} + \sqrt{z} \leq \sqrt{(n+1)(y + y + \cdots + y + z)} =$$
$$= \sqrt{(n+1)(ny+z)} \to \sum_{cyc} \frac{x}{n\sqrt{y} + \sqrt{z}} \geq \sum_{cyc} \frac{x}{\sqrt{(n+1)(ny+z)}}; (1)$$

Applying Weighted Jensen Inequality for $f: (0, \infty) \to \mathbb{R}, f(x) = \frac{1}{\sqrt{x}}$ convex function, we get:

$$\frac{pf(a) + qf(b) + rf(c)}{p+q+r} \geq f\left(\frac{pa + qb + rc}{q+r}\right), p = \frac{x}{3}, q = \frac{y}{3}, r = \frac{z}{3}, p + q + r = 1$$
$$a = (n+1)(ny+z), b = (n+1)(nz+x), c = (n+1)(nx+y) \to$$
$$\sum_{cyc} \frac{x}{3} f\big((n+1)(ny+z)\big) \geq f\left(\sum_{cyc} \frac{x}{3}(n+1)(ny+z)\right) \Leftrightarrow$$
$$\sum_{cyc} \frac{x}{3} \cdot \frac{1}{\sqrt{(n+1)(ny+z)}} \geq \frac{1}{\sqrt{\sum \frac{x}{3}(n+1)(ny+z)}} \Leftrightarrow$$
$$\sum_{cyc} \frac{x}{3} \cdot \frac{1}{\sqrt{(n+1)(ny+z)}} \geq \frac{1}{\sqrt{\frac{1}{3}(n+1)^2(xy+yz+zx)}} \Leftrightarrow$$

$$\sum_{cyc}\frac{x}{\sqrt{(n+1)(ny+z)}}\geq\frac{3\sqrt{3}}{\sqrt{(n+1)^2(xy+yz+zx)}}\Leftrightarrow$$

$$\sum_{cyc}\frac{x}{\sqrt{(n+1)(ny+z)}}\geq\frac{3\sqrt{3}}{(n+1)\sqrt{(xy+yz+zx)}};(2)$$

From (1),(2) it follows that:

$$\sum_{cyc}\frac{x}{n\sqrt{y}+\sqrt{z}}\geq\sum_{cyc}\frac{x}{\sqrt{(n+1)(ny+z)}}\geq\frac{3\sqrt{3}}{(n+1)\sqrt{xy+yz+zx}}$$

Equality holds if and only if $x=y=z=1$.

Lemma 2. If $x,y,z>0, x+y+z=3$ then $\prod(y+z)\geq\frac{8}{3}(xy+yz+zx)$

Proof. Using inequality: **If $x,y,z>0$ then $9\prod(x+y)\geq 8(x+y+z)(xy+yz+zx)$**

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$$9(x+y)(y+z)(z+x)\geq 8(x+y+z)(xy+yz+zx)\Leftrightarrow$$

$$9\left(\sum_{cyc}yz(y+z)+2xyz\right)\geq\sum_{cyc}yz(y+z)+3xyz\Leftrightarrow\sum_{cyc}yz(y+z)\geq 6xyz\Leftrightarrow$$

$\sum_{cyc}x(y-z)^2\geq 0$ true. Equality holds if and only if $x=y=z$.

$$\rightarrow\prod_{cyc}(y+z)\geq\frac{8}{9}(x+y+z)(xy+yz+zx)=\frac{8}{9}\cdot 3(xy+yz+zx)=$$

$=\frac{8}{3}(xy+yz+zx)$. Let's get back to the main problem.

Using Lemma 1,2 we get: $P=\sum_{cyc}\frac{x}{n\sqrt{y}+\sqrt{z}}+\frac{3}{16(n+1)}\prod_{cyc}(y+z)\geq$

$$\geq\frac{3\sqrt{3}}{(n+1)\sqrt{xy+yz+zx}}+\frac{3}{16(n+1)}\cdot\frac{8}{3}(xy+yz+zx)=$$

$$=\frac{3\sqrt{3}}{(n+1)\sqrt{xy+yz+zx}}+\frac{1}{2(n+1)}(xy+yz+zx)\overset{t=\sqrt{\sum yz}}{=}\frac{3\sqrt{3}}{(n+1)t}+\frac{t^2}{2(n+1)}=$$

$$=\frac{3\sqrt{3}}{2(n+1)t}+\frac{3\sqrt{3}}{2(n+1)t}+\frac{t^2}{2(n+1)}\overset{AM-GM}{\geq}3\sqrt[3]{\frac{3\sqrt{3}}{2(n+1)t}\cdot\frac{3\sqrt{3}}{2(n+1)t}\cdot\frac{t^2}{2(n+1)}}=$$

$$=3\cdot\sqrt[3]{\frac{27}{8(n+1)^3}}=\frac{9}{2(n+1)}$$

Therefore, $min\{P\}=\frac{9}{2(n+1)}$, when $x=y=z=1$.

REFERENCE:

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