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ABOUT AN INEQUALITY BY HAXVERDIYEV TAVERDI-V

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1) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{(b^3+c^3)^2}{b^4+c^4}\le 18R^2$$

Proposed by Haxverdiyev Taverdi-Azerbaijan

Solution. Lemma. 2) If $x,y>0$, then $\frac{(x^3+y^3)^2}{x^4+y^4}\le x^2+y^2$

Proof. Using BCS inequality, we get: $\frac{(x^3+y^3)^2}{x^4+y^4}\le x^2+y^2 \Leftrightarrow$

$(x^4+y^4)(x^2+y^2)\overset{CBS}{\ge}(x^3+y^3)^2$. Equality holds if $x=y$.

Let's get back to the main problem. Using Lemma, we get:

$$LHS=\sum_{cyc}\frac{(b^3+c^3)^2}{b^4+c^4}\overset{Lemma}{\le}\sum_{cyc}(b^2+c^2)=2\sum_{cyc}a^2\overset{Leibniz}{\le}2\cdot 9R^2=18R^2=RHS.$$

Equality holds if and only if triangle is equilateral. **Remark.** The problem ca be developed.

3) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{(b^{n+1}+c^{n+1})^2}{b^{2n}+c^{2n}}\le 18R^2, n\in N$$

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Solution. Lemma. 4) If $x,y>0$ and $n\in N$ then: $\frac{(x^{n+1}+y^{n+1})^2}{x^{2n}+y^{2n}}\le x^2+y^2$

Proof. Using CBS inequality, we get:

$$\frac{(x^{n+1}+y^{n+1})^2}{x^{2n}+y^{2n}}\le x^2+y^2 \Leftrightarrow (x^{2n}+y^{2n})(x^2+y^2)\overset{CBS}{\ge}(x^{n+1}+y^{n+1})^2$$

Equality holds if $x=y$. Let's get back to the main problem.

Using Lemma, we get:

$$LHS=\sum_{cyc}\frac{(b^{n+1}+c^{n+1})^2}{b^{2n}+c^{2n}}\overset{Lemma}{\le}\sum_{cyc}(b^2+c^2)=2\sum_{cyc}a^2\overset{Leibniz}{\le}2\cdot 9R^2=18R^2=RHS.$$

Equality holds if and only if triangle is equilateral.

Note. For $n=2$ we get proposed problem by Haxverdiyev Taverdi-Azerbaijan.

5) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\left(h_b^{n+1}+h_c^{n+1}\right)^2}{h_b^{2n}+h_c^{2n}}\le\frac{27}{2}R^2, n\in N$$

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Solution. Lemma. 6) If $x,y>0$ and $n\in N$ then: $\frac{\left(x^{n+1}+y^{n+1}\right)^2}{x^{2n}+y^{2n}}\le x^2+y^2$

Proof. Using CBS inequality, we get:

$$\frac{(x^{n+1}+y^{n+1})^2}{x^{2n}+y^{2n}}\le x^2+y^2\Leftrightarrow(x^{2n}+y^{2n})(x^2+y^2)\overset{CBS}{\ge}(x^{n+1}+y^{n+1})^2$$

Equality holds if $x=y$. Let's get back to the main problem.

Using Lemma, we get:

$$LHS=\sum_{cyc}\frac{(h_b^{n+1}+h_c^{n+1})^2}{h_b^{2n}+h_c^{2n}}\overset{Lemma}{\le}\sum_{cyc}(h_b^2+h_c^2)=2\sum_{cyc}h_a^2=2\sum_{cyc}\left(\frac{2F}{a}\right)^2=$$

$$=2\cdot 4F^2\sum_{cyc}\frac{1}{a^2}\overset{Steing}{\le}2\cdot 4s^2r^2\cdot\frac{1}{4r^2}=2s^2\overset{Mitrinovic}{\le}2\cdot\frac{27R^2}{4}=\frac{27R^2}{2}=RHS$$

Equality holds if and only if triangle is equilateral.

7) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\left(r_b^{n+1}+r_c^{n+1}\right)^2}{r_b^{2n}+r_c^{2n}}\le\frac{27R^3}{4r}, n\in N$$

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Solution. Lemma. 8) If $x,y>0$ and $n\in N$ then: $\frac{\left(x^{n+1}+y^{n+1}\right)^2}{x^{2n}+y^{2n}}\le x^2+y^2$

Proof. Using CBS inequality, we get:

$$\frac{(x^{n+1}+y^{n+1})^2}{x^{2n}+y^{2n}}\le x^2+y^2\Leftrightarrow(x^{2n}+y^{2n})(x^2+y^2)\overset{CBS}{\ge}(x^{n+1}+y^{n+1})^2$$

Equality holds if $x=y$. Let's get back to the main problem.

Using Lemma, we get:

$$LHS=\sum_{cyc}\frac{(r_b^{n+1}+r_c^{n+1})^2}{r_b^{2n}+r_c^{2n}}\overset{Lemma}{\le}\sum_{cyc}(r_b^2+r_c^2)=2\sum_{cyc}r_a^2=2F^2\sum_{cyc}\frac{1}{(s-a)^2}=$$

$$=2s^2\left[\left(\frac{4R+r}{s}\right)^2-2\right]=2[(4R+r)^2-2s^2]\overset{Gerretsen}{\le}2[(4R+r)^2-2(16Rr-5r^2)]$$

$$=1(16R^2-24Rr+11r^2)\overset{Euler}{\le}\frac{27R^3}{4r}=RHS$$

Equality holds if and only if triangle is equilateral.

9) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\left(m_b^{n+1}+m_c^{n+1}\right)^2}{m_b^{2n}+m_c^{2n}}\le\frac{27}{2}R^2, n\in N$$

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Solution. Lemma. 10) If $x,y>0$ and $n\in N$ then: $\frac{(x^{n+1}+y^{n+1})^2}{x^{2n}+y^{2n}}\le x^2+y^2$

Proof. Using CBS inequality, we get:

$$\frac{(x^{n+1}+y^{n+1})^2}{x^{2n}+y^{2n}}\le x^2+y^2\Leftrightarrow(x^{2n}+y^{2n})(x^2+y^2)\overset{CBS}{\ge}(x^{n+1}+y^{n+1})^2$$

Equality holds if $x=y$. Let's get back to the main problem.

Using Lemma, we get:

$$\sum_{cyc}\frac{(m_b^{n+1}+m_c^{n+1})^2}{m_b^{2n}+m_c^{2n}}\le\sum_{cyc}(m_b^2+m_c^2)=2\sum_{cyc}m_a^2=2\cdot\frac{3}{4}\sum_{cyc}a^2\overset{Leibniz}{\le}2\cdot\frac{3}{4}\cdot 9R^2=\frac{27}{2}R^2$$
$$=RHS.$$

11) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\left(s_b^{n+1}+s_c^{n+1}\right)^2}{s_b^{2n}+s_c^{2n}}\le\frac{27}{2}R^2, n\in N$$

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Solution. Lemma. 12) If $x,y>0$ and $n\in N$ then: $\frac{(x^{n+1}+y^{n+1})^2}{x^{2n}+y^{2n}}\le x^2+y^2$

Proof. Using CBS inequality, we get:

$$\frac{(x^{n+1}+y^{n+1})^2}{x^{2n}+y^{2n}}\le x^2+y^2\Leftrightarrow(x^{2n}+y^{2n})(x^2+y^2)\overset{CBS}{\ge}(x^{n+1}+y^{n+1})^2$$

Equality holds if $x=y$. Let's get back to the main problem.

Using Lemma, we get:

$$\sum_{cyc}\frac{(s_b^{n+1}+s_c^{n+1})^2}{s_b^{2n}+s_c^{2n}}\le\sum_{cyc}(s_b^2+s_c^2)=2\sum_{cyc}s_a^2\le 2\sum_{cyc}m_a^2=2\cdot\frac{3}{4}\sum_{cyc}a^2\overset{Leibniz}{\le}2\cdot\frac{3}{4}\cdot 9R^2$$
$$=\frac{27}{2}R^2=RHS.$$

Reference:

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