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ABOUT AN INEQUALITY BY ADIL ABDULLAYEV-XX

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1) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{b+c}{\sqrt{b^2+c^2}}\geq 6\sqrt{\frac{R}{2}}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution. **Lemma(Băndilă) 2) In ΔABC the following relationship holds:**

$$\frac{R}{r}\geq\frac{b}{c}+\frac{c}{b}$$

Proof. Using Ravi substitution: $a=y+z, b=z+x, c=x+y$, we get:

$$R=\frac{(x+y)(y+z)(z+x)}{4\sqrt{xyz(x+y+z)}}, r=\sqrt{\frac{xyz}{x+y+z}}\to\frac{R}{r}=\frac{(x+y)(y+z)(z+x)}{xyz}$$

Inequality can be written as: $\frac{(x+y)(y+z)(z+x)}{xyz}\geq\frac{z+x}{x+y}+\frac{x+y}{z+x}\Leftrightarrow$

$\frac{y+z}{4xyz}\geq\frac{1}{(x+y)^2}+\frac{1}{(x+z)^2}$ which is true from: $\begin{cases}\frac{1}{(x+y)^2}\leq\frac{1}{4xy}\\ \frac{1}{(x+z)^2}\leq\frac{1}{4xz}\end{cases}$. Equality holds for $x=z$.

Equality if and only if triangle is equilateral. Let's get back to the main problem.

Using $\frac{R}{r}\geq\frac{b}{c}+\frac{c}{b}\to\frac{bc}{b^2+c^2}\geq\frac{r}{R}$. Using AM-GM inequality, we get:

$$LHS=\sum_{cyc}\frac{b+c}{\sqrt{b^2+c^2}}\geq\sum_{cyc}\frac{2\sqrt{bc}}{\sqrt{b^2+c^2}}=2\sum_{cyc}\sqrt{\frac{bc}{b^2+c^2}}\geq$$

$$\geq 2\sum_{cyc}\sqrt{\frac{r}{R}}=2\cdot 3\sqrt{\frac{r}{R}}=6\sqrt{\frac{r}{R}}=RHS$$

Equality holds if and only if triangle is equilateral.

3) In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{bc}{b^2+c^2}} \geq 3\sqrt{\frac{r}{R}}$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma(Băndilă) 2) In ΔABC the following relationship holds:**

$$\frac{R}{r} \geq \frac{b}{c}+\frac{c}{b}$$

Proof. Using Ravi substitution: $a=y+z, b=z+x, c=x+y$, we get:

$$R=\frac{(x+y)(y+z)(z+x)}{4\sqrt{xyz(x+y+z)}}, r=\sqrt{\frac{xyz}{x+y+z}} \rightarrow \frac{R}{r}=\frac{(x+y)(y+z)(z+x)}{xyz}$$

Inequality can be written as: $\frac{(x+y)(y+z)(z+x)}{xyz} \geq \frac{z+x}{x+y}+\frac{x+y}{z+x} \Leftrightarrow$

$\frac{y+z}{4xyz} \geq \frac{1}{(x+y)^2}+\frac{1}{(x+z)^2}$ which is true from: $\begin{cases}\frac{1}{(x+y)^2} \leq \frac{1}{4xy} \\ \frac{1}{(x+z)^2} \leq \frac{1}{4xz}\end{cases}$. Equality holds for $x=z$.

Equality if and only if triangle is equilateral. Let's get back to the main problem.

Using $\frac{R}{r} \geq \frac{b}{c}+\frac{c}{b} \rightarrow \frac{bc}{b^2+c^2} \geq \frac{r}{R}$. Using AM-GM inequality, we get:

$LHS=\sum_{cyc}\sqrt{\frac{bc}{b^2+c^2}} \geq \sum_{cyc}\sqrt{\frac{r}{R}}=3\sqrt{\frac{r}{R}}=RHS$. Equality holds if and only if triangle is equilateral.

5) In ΔABC the following relationship holds:

$$3\sqrt{\frac{r}{R}} \leq \sum_{cyc}\sqrt{\frac{bc}{b^2+c^2}}=\frac{3}{\sqrt{2}}$$

Proposed by Marin Chirciu-Romania

Solution. For RHS using AM-GM inequality, we have: $\frac{bc}{b^2+c^2} \leq \frac{1}{2}$, equality when $b=c$.

For LHS, **Lemma(Băndilă) 2) In ΔABC the following relationship holds:**

$$\boldsymbol{\frac{R}{r} \geq \frac{b}{c} + \frac{c}{b}}$$

Proof. Using Ravi substitution: $a = y + z, b = z + x, c = x + y$, we get:

$$R = \frac{(x+y)(y+z)(z+x)}{4\sqrt{xyz(x+y+z)}}, r = \sqrt{\frac{xyz}{x+y+z}} \rightarrow \frac{R}{r} = \frac{(x+y)(y+z)(z+x)}{xyz}$$

Inequality can be written as: $\frac{(x+y)(y+z)(z+x)}{xyz} \geq \frac{z+x}{x+y} + \frac{x+y}{z+x} \Leftrightarrow$

$\frac{y+z}{4xyz} \geq \frac{1}{(x+y)^2} + \frac{1}{(x+z)^2}$ which is true from: $\begin{cases} \frac{1}{(x+y)^2} \leq \frac{1}{4xy} \\ \frac{1}{(x+z)^2} \leq \frac{1}{4xz} \end{cases}$. Equality holds for $x = z$.

Equality if and only if triangle is equilateral. Let's get back to the main problem.

Using $\frac{R}{r} \geq \frac{b}{c} + \frac{c}{b} \rightarrow \frac{bc}{b^2+c^2} \geq \frac{r}{R}$. Using AM-GM inequality, we get:

$$LHS = \sum_{cyc} \sqrt{\frac{bc}{b^2+c^2}} \geq \sum_{cyc} \sqrt{\frac{r}{R}} = 3\sqrt{\frac{r}{R}} = RHS.$$

Equality holds if and only if triangle is equilateral.

REFERENCE:

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