ABOUT A FAMOUS INEQUALITY

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A famous inequality is A.M. Nesbitt-I. Ionescu inequality:

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geq\frac{3}{2},\forall x,y,z\in\mathbb{R}_+^*=(0,\infty) \tag{N-I}$$

Proof 1. We have:

$$\sum_{cyc}\frac{x}{y+z}=-3+\sum_{cyc}\Big(\frac{x}{y+z}+1\Big)=-3+(x+y+z)\cdot\sum_{cyc}\frac{1}{y+z}\overset{\text{Bergström}}{\geq}$$

$$\geq-3+(x+y+z)\cdot\frac{(1+1+1)^2}{\sum_{cyc}(y+z)}=-3+(x+y+z)\cdot\frac{9}{2(x+y+z)}=\frac{9}{2}-3=\frac{3}{2}$$

□

Proof 2. We have:

$$\sum_{cyc}\frac{x}{y+z}=\sum_{cyc}\frac{x^2}{x(y+z)}=\sum_{cyc}\frac{x^2}{xy+xz}\overset{\text{Bergström}}{\geq}\frac{(x+y+z)^2}{\sum_{cyc}(xy+xz)}=\frac{(x+y+z)^2}{2(xy+yz+zx)}\geq$$

$$\geq\frac{3(xy+yz+zx)}{2(xy+yz+zx)}=\frac{3}{2}$$

□

Proof 3. Let $X=x+y+z$, then:

$$\sum_{cyc}\frac{x}{y+z}=\sum_{cyc}\frac{x}{X-x}=\sum_{cyc}\frac{x^2}{X\cdot x-x^2}\overset{\text{Bergström}}{\geq}\frac{(x+y+z)^2}{\sum_{cyc}(X\cdot x-x^2)}=$$

$$=\frac{X^2}{X\cdot\sum_{cyc}x-\sum_{cyc}x^2}=\frac{X^2}{X^2-\sum_{cyc}x^2}\geq\frac{X^2}{X^2-\frac{1}{3}X^2}=\frac{3}{3-1}=\frac{3}{2}$$

□

Now, let's prove some inequalities about this interesting, beautiful and famous inequality.

1.
$$\frac{x}{(y+z)^2}+\frac{y}{(z+x)^2}+\frac{z}{(x+y)^2}\geq\frac{9}{4(x+y+z)},\forall x,y,z\in\mathbb{R}_+^* \tag{1}$$

Proof 1. We have:

$$\sum_{cyc}\frac{x}{(y+z)^2}=\sum_{cyc}\frac{\left(\frac{x}{y+z}\right)^2}{x}\overset{\text{Bergström}}{\geq}\frac{\left(\sum_{cyc}\frac{x}{y+z}\right)^2}{\sum_{cyc}x}\overset{\text{(N-I)}}{\geq}\frac{\frac{9}{4}}{x+y+z}=\frac{9}{4(x+y+z)}$$

□

Proof 2. We have:

$$\sum_{cyc}\frac{x}{(y+z)^2}=\sum_{cyc}\frac{x^3}{(xy+yz)^2}\overset{\text{J. Radon}}{\geq}\frac{(x+y+z)^3}{(\sum_{cyc}(xy+xz))^2}=\frac{X^3}{4(xy+yz+zx)^2}\geq$$

$$\geq\frac{X^3}{4(\frac{(x+y+z)^2}{3})^2}=\frac{X^3}{4\cdot\frac{X^4}{9}}=\frac{9X^3}{4X^4}=\frac{9}{4X}=\frac{9}{4(x+y+z)}$$

□

(B-G) **2.** $\frac{x}{(y+z)^3}+\frac{y}{(z+x)^3}+\frac{z}{(x+y)^3}\geq\frac{27}{8(x+y+z)^2};\forall x,y,z\in\mathbb{R}_+^*$

This inequality is a problem proposed by D.M. Bătineţu-Giurgiu, in R.M.M, and in [1] this inequality was named Bătineţu inequality. We propose to be named (B-G) namely Bătineţu-Giurgiu inequality.

Proof 1. We have:

$$\sum_{cyc}\frac{x}{(y+z)^3}=\sum_{cyc}\frac{x^4}{(xy+xz)^3}\overset{\text{J. Radon}}{\geq}\frac{(x+y+z)^4}{(\sum_{cyc}(xy+xz))^3}=\frac{X^4}{8(xy+yz+zx)^3}\geq$$

$$\geq\frac{X^4}{8(\frac{(x+y+z)^2}{3})^3}=\frac{27X^4}{8X^6}=\frac{27}{8X^2}=\frac{27}{8(x+y+z)^2}$$

□

Proof 2. We have:

$$\sum_{cyc}\frac{x}{(y+z)^3}=\sum_{cyc}\frac{(\frac{x}{y+z})^3}{x^2}\overset{\text{J. Radon}}{\geq}\frac{(\sum_{cyc}\frac{x}{y+z})^3}{(x+y+z)^2}\overset{\text{(N-I)}}{\geq}\frac{(\frac{3}{2})}{X^2}=\frac{27}{8X^2}=\frac{27}{8(x+y+z)^2}$$

□

3. If $m\in\mathbb{R}_+=[0,\infty)$ and $x,y,z\in\mathbb{R}_+^*$, then:

$$\frac{x}{(y+z)^{m+1}}+\frac{y}{(z+x)^{m+1}}+\frac{z}{(x+y)^{m+1}}\geq\frac{3^{m+1}}{2^{m+1}(x+y+z)^m}\tag{3}$$

Proof 1. We have:

$$\sum_{cyc}\frac{x}{(y+z)^{m+1}}=\sum_{cyc}\frac{x^{m+2}}{(xy+xz)^{m+1}}\overset{\text{J. Radon}}{\geq}\frac{(x+y+z)^{m+2}}{(\sum_{cyc}(xy+xz)^{m+1}}=\frac{X^{m+2}}{2^{m+1}(xy+yz+zx)^{m+1}}\geq$$

$$\geq\frac{X^{m+2}}{2^{m+1}(\frac{(x+y+z)^2}{3})^{m+1}}=\frac{3^{m+1}\cdot X^{m+2}}{2^{m+1}\cdot X^{2m+2}}=\frac{3^{m+1}}{2^{m+1}\cdot X^m}=\frac{3^{m+1}}{2^{m+1}(x+y+z)^m}$$

□

Proof 2. We have:

$$\sum_{cyc}\frac{x}{(y+z)^{m+1}}=\sum_{cyc}\frac{(\frac{x}{y+z})^{m+1}}{x^m}\overset{\text{J. Radon}}{\geq}\frac{(\sum_{cyc}\frac{x}{y+z})^{m+1}}{(x+y+z)^m}\geq$$

$$\overset{\text{(N-I)}}{\geq}\frac{(\frac{3}{2})^{m+1}}{X^m}=\frac{3^{m+1}}{2^{m+1}X^m}=\frac{3^{m+1}}{2^{m+1}(x+y+z)^m}$$

Observation: If $m = 0$ then from relationship (3) we obtain inequality (N-I), and if $m = 2$ from relationship (3) we obtain Bătineţu - Giurgiu inequality. □

4. Let be $n \in \mathbb{N}, n \geq 3, x_k \in \mathbb{R}_+^*, \forall k = \overline{1,n}$ and $X_n = \sum_{k=1}^n x_k$, then:

$$\sum_{k=1}^n \frac{x_k}{(X_n - x_k)^{m+1}} \geq \frac{n^{m+1}}{(n-1)^{m+1} \cdot X_n^m}, \forall m \in \mathbb{R}_+ \tag{4}$$

Proof 1. We have:

$$\sum_{k=1}^n \frac{x_k}{(X_n - x_k)^{m+1}} = \sum_{k=1}^n \frac{\left(\frac{x_k}{X_n - x_k}\right)^{m+1}}{x_k^m} \overset{\text{J. Radon}}{\geq} \frac{\left(\sum_{k=1}^n \frac{x_k}{X_n - x_k}\right)^{m+1}}{\left(\sum_{k=1}^n x_k\right)^m} =$$

$$= \frac{1}{X^m} \left(\sum_{k=1}^n \frac{x_k}{X_n - x_k} \right)^m \tag{5}$$

We have:

$$\sum_{k=1}^n \frac{x_k}{X_n - x_k} = -n + \sum_{k=1}^n \left(\frac{x_k}{X_n - x_k} + 1 \right) = -n + X_n \cdot \sum_{k=1}^n \frac{1}{X_n - x_k} \geq$$

$$\geq -n + X_n \cdot \frac{(\underbrace{1 + 1 + \ldots + 1 + 1}_{n \text{ times}})^2}{\sum_{k=1}^n (X_n - x_k)} = -n + X_n \cdot \frac{n^2}{nX_n - \sum_{k=1}^n x_k} =$$

$$= -n + \frac{n^2 X_n}{(n-1)X_n} = -n + \frac{n^2}{n-1} = \frac{n}{n-1} \tag{6}$$

From relationships (5) and (6) we deduce that:

$$\sum_{k=1}^n \frac{x_k}{(X_n - x_k)^{m+1}} \geq \frac{n^{m+1}}{(n-1)^{m+1} X_n^m} = \frac{n^{m+1}}{(n-1)^{m+1}(x_1 + x_2 + \ldots + x_n)^m}$$

□

Proof 2. We have:

$$\sum_{k=1}^n \frac{x_k}{(X_n - x_k)^{m+1}} = \sum_{k=1}^n \frac{x_k^{m+2}}{(X_n \cdot x_k - x_k^2)^{m+1}} \overset{\text{J. Radon}}{\geq} \frac{\left(\sum_{k=1}^n\right)^{m+2}}{\left(\sum_{k=1}^n (X_n \cdot x_k - x_k^2)\right)^{m+1}} =$$

$$= \frac{X_n^{m+2}}{(X_n^2 - \sum_{k=1}^n x_k^2)^{m+1}} \geq \frac{X_n^{m+2}}{(X_n^2 - \frac{1}{n}(\sum_{k=1}^n x_k)^2)^{m+1}} = \frac{n^{m+1} X_n^{m+2}}{(nX_n^2 - X_n^2)^{m+1}} =$$

$$= \frac{n^{m+1} \cdot X_n^{m+2}}{(n-1)^{m+1} \cdot X_n^{2m+2}} = \frac{n^{m+1}}{(n-1)^{m+1} \cdot X_n^m} = \frac{n^{m+1}}{(n-1)^{m+1}(x_1 + x_2 + \ldots + x_n)^m}$$

Observation. For $n = 3$ from relationship (4) we obtain relationship (3). □

Applications:

A1. If $x, y, z > 0$ and ABC is a triangle with the area F, then:

$$\frac{x \cdot a^4}{(y+z)^3} + \frac{y \cdot b^4}{(z+x)^3} + \frac{zc^4}{(x+y)^3} \geq \frac{18F^2}{(x+y+z)^2} \tag{5}$$

Proof. We have:

$$\sum_{cyc}\frac{xa^4}{(y+z)^3}=\sum_{cyc}\frac{\frac{x^2a^4}{(y+z)^2}}{x(y+z)}=\sum_{cyc}\frac{\left(\frac{xa^2}{y+z}\right)^2}{xy+xz}\overset{\text{Bergström}}{\geq}$$

$$\geq\frac{\left(\sum_{cyc}\frac{xa^2}{y+z}\right)^2}{\sum_{cyc}(xy+xz)}\overset{\text{Tsintsifas}}{\geq}\frac{(2\sqrt{3}F)^2}{2\sum_{cyc}xy}=\frac{6F^2}{xy+yz+zx}\geq$$

$$\geq\frac{6F^2}{\frac{(x+y+z)^2}{3}}=\frac{18F^2}{(x+y+z)^2}$$

□

A2. If $x,y,z>0$ then in ΔABC with the area F the following inequality holds:

$$\frac{xa^8}{(y+z)^3}+\frac{yb^8}{(z+x)^3}+\frac{zc^8}{(x+y)^3}\geq\frac{6F^4}{(x+y+z)^2} \tag{6}$$

Proof. We have:

$$\sum_{cyc}\frac{xa^8}{(y+z)^3}=\sum_{cyc}\frac{(xa^2)^4}{(xy+xz)^3}\overset{\text{J.Radon}}{\geq}\frac{(xa^2+yb^2+zc^2)^4}{\left(\sum_{cyc}(xy+xz)\right)^3}=$$

$$=\frac{(xa^2+yb^2+zc^2)^4}{(2(xy+yz+zx))^3}=\frac{1}{8}\cdot\frac{(xa^2+yb^2+zc^2)^4}{(xy+yz+zx)^3}\geq$$

$$\overset{\text{Klamkin}}{\geq}\frac{(4\sqrt{xy+yz+zx}F)^4}{8(xy+yz+zx)^3}=\frac{2(xy+yz+zx)^2F^4}{(xy+yz+zx)^3}=$$

$$=\frac{2F^4}{xy+yz+zx}\geq\frac{2F^4}{\frac{(x+y+z)^2}{3}}=\frac{6F^4}{(x+y+z)^2}$$

□

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