A BEAUTIFUL EQUIVALENCE

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Abstract. In this paper we prove the equivalence of Gerretsen's inequality in triangle and the algebraic Schur's inequality.

Gerretsen's inequality: In ΔABC the following relationship holds:
$s^2 \geq 16Rr - 5r^2$ (s - semiperimeter, r - in radii, R - circumradii)
Schur's inequality: If $x, y, z > 0$ then:

$$x^3 + y^3 + z^3 + 3xyz \geq xy(x+y) + yz(y+z) + zx(z+x)$$

The proof of equivalence:

$$s^2 \geq 16Rr - 5r^2 \text{ (Gerretsen) } \to s^2 + r^2 + 4r^2 \geq 16Rr \to$$

$$\frac{s^2 + r^2}{4Rr} + \frac{r}{R} \geq 4 \to \frac{s^2 + r^2}{4Rr} + 1 + \frac{r}{R} \geq 5 \to$$

$$\frac{s^2 + r^2 + 4Rr}{4Rr} + \frac{r}{R} \geq 5 \to \frac{ab + bc + ca}{4Rr} + \frac{r}{R} \geq 5 \to$$

$$\frac{s(ab + bc + ca)}{4Rrs} + \frac{r}{R} \geq 5 \to \frac{s(ab + bc + ca)}{4RF} + \frac{r}{R} \geq 5 \to$$

$$s \cdot \frac{ab + bc + ca}{abc} + \frac{r}{R} \geq 5 \to s\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + \frac{r}{R} \geq 5 \to$$

$$a = y + z, b = z + x, c = x + y, s = x + y + z, \frac{r}{R} = \frac{4xyz}{(x+y)(y+z)(z+x)}$$

$$(x + y + z)\left(\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+y}\right) + \frac{4xyz}{(x+y)(y+z)(z+x)} \geq 5$$

$$(x+y+z)\frac{(x+y)(y+z) + (y+z)(z+x) + (z+x)(x+y)}{(x+y)(y+z)(z+x)} + \frac{4xyz}{(x+y)(y+z)(z+x)} \geq 5$$

$$(x + y + z)[(x+y)(y+z) + (y+z)(z+x) + (z+x)(x+y)] + 4xyz \geq$$

$$\geq 5(x+y)(y+z)(z+x)$$

Let $p = x + y + z, q = xy + yz + zx, r = xyz$

$$(1) \Leftrightarrow p(p^2 + q) + 4r \geq 5(pq - r) \Leftrightarrow p^3 + 9r \geq 4pq$$

$$\Leftrightarrow p^3 - 3pq + 3r + 3r \geq pq - 3r$$

$$x^3 + y^3 + z^3 = p^3 - 3pq + 3r$$

$$xy(x+y) + yz(y+z) + zx(z+x) = pq - 3r$$

$$x^3 + y^3 + z^3 + 3xyz \geq xy(x+y) + yz(y+z) + zx(z+x)$$

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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