Some interesting properties of complex numbers with the same modulus

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Abstract. In this paper we will see some less known theorems concerning complex numbers in Geometry, which allow us to find easier proofs, if we use the affixes of some points of the unit circle.

Keywords: complex number, incircle, circumcircle, projection, collinearity, concurrency.

1. Theoretical approach

In this paper we will see some interesting properties of complex numbers in Geometry which, together with a convenient choice of the axes of the complex plane, allow us to find easier proofs for difficult problems. If we consider the affixes of some points which lie on the unit circle, the proofs may become shorter and nicer.

We will use the notions and the assertions presented in [1]. If M is a point of the complex plane, we denote by m its affix. In this paper we consider that the unit circle is centered at the origin $O\left(0\right)$ of the complex plane.

Theorem 1. *If $A\left(a\right)$, $B\left(b\right)$ are two points of the complex plane, the equation of the line which contains the point $M\left(m\right)$ and is perpendicular to the line AB is:*

$$z\left(\overline{b}-\overline{a}\right)+\overline{z}\left(b-a\right)=m\left(\overline{b}-\overline{a}\right)+\overline{m}\left(b-a\right).$$

Proof. The equation of the line is:

$$\frac{z-m}{a-b}=\frac{\overline{z}-\overline{m}}{\overline{b}-\overline{a}}\Leftrightarrow z\left(\overline{b}-\overline{a}\right)+\overline{z}\left(b-a\right)=m\left(\overline{b}-\overline{a}\right)+\overline{m}\left(b-a\right).$$

Theorem 2. *We consider a triangle ABC, the center $O(o)$ of its circumscribed circle, its orthocenter $H(h)$, its center of gravity $G(g)$ and a point M of the complex plane.*

a) $\quad o = \dfrac{|a|^2(b-c)+|b|^2(c-a)+|c|^2(a-b)}{\bar{a}(b-c)+\bar{b}(c-a)+\bar{c}(a-b)}$

b) $\quad h = \dfrac{\sum a^2(\bar{c}-\bar{b})+\sum|a|^2(c-b)}{\sum \bar{a}(b-c)}.$

c) $\quad h + 2o = 3g = a+b+c.$

d) $\quad AM$ *is the bisector line of* $\widehat{BAC}$ *iff* $\dfrac{(m-a)^2}{(b-a)(c-a)} \in \mathbb{R}_+$.

e) $\quad AN$ *is the outer bisector line of* $\widehat{BAC}$ *iff* $\dfrac{(n-a)^2}{(b-a)(c-a)} \in \mathbb{R}_-.$

Proof. **a)** By replacing $m = \frac{a+b}{2}$ in **Theorem 1**, we deduce that the mediator line of the segment AB has the equation:

$$z(\bar{b}-\bar{a}) + \bar{z}(b-a) + |a|^2 - |b|^2 = 0.$$

Similarly, the mediator line of the segment AC has the equation:

$$z(\bar{c}-\bar{a}) + \bar{z}(c-a) + |a|^2 - |c|^2 = 0.$$

By multiplying the first equation with $(c-a)$, the second one with $(b-a)$ and then by substracting the equations thus obtained, the conclusion follows.

b) Using **Theorem 1**, we find the equations of the heights from A and B of the triangle:

$$h_A : z(\bar{c}-\bar{b}) + \bar{z}(c-b) = a(\bar{c}-\bar{b}) + \bar{a}(c-b),$$

$$h_B : z(\bar{a}-\bar{c}) + \bar{z}(a-c) = b(\bar{a}-\bar{c}) + \bar{b}(a-c).$$

By multiplying the first equation with $(a-c)$, the second one with $(b-c)$ and then by adding the equations thus obtained, we find the conclusion.

c) The relation is a transcription of the fact that O, G and H lie on Euler's line of the triangle ABC.

d) AM is the bisector line of $\widehat{BAC}$ $\Leftrightarrow \arg\left(\frac{m-a}{b-a}\right) = \arg\left(\frac{c-a}{m-a}\right) \Leftrightarrow \dfrac{\frac{m-a}{b-a}}{\frac{c-a}{m-a}} \in \mathbb{R}_+$

$\Leftrightarrow \dfrac{(m-a)^2}{(b-a)(c-a)} \in \mathbb{R}_+.$

e) Let $D \in AC$ be such that $A \in (CD)$.

Then, $k \in \mathbb{R}_+$ exists, such that $d - a = k(a-c)$.

AN is the outer bisector line of $\widehat{BAC}$ iff $\arg\left(\frac{b-a}{n-a}\right) = \arg\left(\frac{n-a}{d-a}\right) \Leftrightarrow$

$\dfrac{(n-a)^2}{(b-a)(d-a)} \in \mathbb{R}_+ \Leftrightarrow -\dfrac{(n-a)^2}{k(b-a)(c-a)} \in \mathbb{R}_+ \Leftrightarrow \dfrac{(n-a)^2}{(b-a)(c-a)} \in \mathbb{R}_-.$

Remark.

i) If O is the origin of the complex plane, then from any of the assertions **b**) and **c**) it follows that $h = a + b + c$.

ii) If the point C is the origin of the complex plane, then

$$o = \frac{ab\left(\bar{a} - \bar{b}\right)}{\bar{a}b - a\bar{b}}, \quad h = \frac{\left(\bar{a}b + a\bar{b}\right)(a - b)}{a\bar{b} - \bar{a}b}.$$

Theorem 3. *The points A and B lie on the unit circle, and the points C and Z are arbitrary. We have:*

a) $\quad Z \in AB \iff z + ab\bar{z} = a + b$.

b) $\quad Z$ *lies on the tangent line at A to the circle* $\iff z + a^2\bar{z} = 2a$.

c) $\quad Z$ *lies on the perpendicular line through C on AB* $\iff z - ab\bar{z} = c - ab\bar{c}$.

d) $\quad Z$ *lies on the line through C which is orthogonal to the tangent line at A to the circle* $\iff z - a^2\bar{z} = c - a^2\bar{c}$.

Proof.

$$\textbf{a) } Z \in AB \iff \frac{z - a}{b - a} \in \mathbb{R} \iff \frac{z - a}{b - a} = \frac{\bar{z} - \bar{a}}{\bar{b} - \bar{a}} \iff z\bar{b} - z\bar{a} - a\bar{b} = b\bar{z} - a\bar{z} - \bar{a}b.$$

By multiplying the previous equality with ab and using $a\bar{a} = b\bar{b} = 1$, we deduce:

$$(a - b)(z + ab\bar{z} - a - b) = 0 \iff z + ab\bar{z} = a + b.$$

b) $\quad \frac{z-a}{0-a} \in i\mathbb{R} \iff \frac{a-z}{a} = \frac{\bar{z}-\bar{a}}{\bar{a}} \iff |a|^2 - \bar{a}z = a\bar{z} - |a|^2 \iff \bar{a}z + a\bar{z} = 2 \overset{|\cdot a}{\iff}$
$\iff z + a^2\bar{z} = 2a$.

c) $\quad \frac{z-c}{a-b} \in i\mathbb{R} \iff \frac{z-c}{a-b} = \frac{\bar{z}-\bar{c}}{\bar{b}-\bar{a}} \iff \left(\bar{b} - \bar{a}\right)z - \bar{b}c + \bar{a}c = (a - b)\bar{z} - a\bar{c} + b\bar{c} \overset{|\cdot ab}{\iff}$
$\iff (a - b)z - c(a - b) = ab(a - b)\bar{z} - ab\bar{c}(a - b) \iff z - ab\bar{z} = c - ab\bar{c}$.

d) $ZC \parallel OA \iff \frac{z-c}{a-0} \in \mathbb{R} \iff \frac{z-c}{a} = \frac{\bar{z}-\bar{c}}{\bar{a}} \iff \bar{a}z - \bar{a}c = a\bar{z} - a\bar{c} \overset{|\cdot a}{\iff} z - a^2\bar{z} = c - a^2\bar{c}$.

Theorem 4. *The points A, B, C and D lie on the unit circle and Z is an arbitrary point. Then:*

a) $\quad \frac{a-b}{\bar{a}-\bar{b}} = -ab$.

b) *The affix of the projection of the point Z on the line AB is* $\quad m = \frac{a+b+z-ab\bar{z}}{2}$.

c) *The affix of the intersection point of the chords AB and CD is*

$$p = \frac{ab(c+d) - cd(a+b)}{ab - cd}.$$

d) *The affix of the intersection point of the tangent lines at A and B to the circle is* $\quad q = \frac{2ab}{a+b}$.

Proof. **a**) $\frac{a-b}{\bar{a}-\bar{b}} = \frac{ab(\bar{a}-\bar{b})}{a\bar{a}b - b\bar{b}a} = \frac{ab(\bar{a}-\bar{b})}{b-a} = -ab$.

b) We consider $M(m) = pr_{AB}(Z)$.

As $M \in AB$, from Theorem 3.a) follows:

$$m + ab\bar{m} = a + b \tag{1.1}$$

Because M lies on the line through Z which is orthogonal to AB, we deduce, using **Theorem 3.c**):

$$m - ab\overline{m} = z - ab\overline{z} \qquad (1.2)$$

By adding the equalities (1.1) and (1.2) it results $\quad m = \frac{a+b+z-ab\overline{z}}{2}$.

c) We consider $\{P\} = AB \cap CD$. **Theorem 1, a)** leads to:

$$p + cd\overline{p} = c + d$$

$$p + ab\overline{p} = a + b$$

By multiplying the first equality with ab, the second one with cd and by substracting the equations thus obtained, we deduce:

$$(ab - cd)\, p = ab\,(c + d) - cd\,(a + b).$$

d) Let $Q\,(q)$ be the intersection point of these tangent lines. Because Q lies on the tangent line at A to the circle, from **Theorem 3.b**) we obtain:

$$q + a^2\overline{q} = 2a$$

But Q also lies on the tangent line at B to the circle, therefore, from **Theorem 3.b**) we deduce:

$$q + b^2\overline{q} = 2b.$$

By multiplying the first equality with b^2, the second one with a^2 and by substracting the equations thus obtained, it results:

$$\left(b^2 - a^2\right) q = 2ab\,(b - a) \iff q = \frac{2ab}{a + b}.$$

Theorem 5. *If the incircle of the triangle ABC is the unit circle and its intersection points with the lines BC, CA and AB are P, Q, R, then:*

a) $\quad a = \frac{2qr}{q+r}, \quad b = \frac{2rp}{r+p}, \quad c = \frac{2pq}{p+q}.$

b) $\quad o = \frac{2pqr\,(p+q+r)}{(p+q)(q+r)(r+p)}.$

c) $\quad h = \frac{2\left(p^2q^2+q^2r^2+r^2p^2+pqr(p+q+r)\right)}{(p+q)(q+r)(r+p)}.$

Proof. **a)** It results from **Theorem 4. d)**.

b) From **Theorem 2. a)**, we have: $\quad o = \frac{\sum |a|^2 (b-c)}{\sum \overline{a}(b-c)}.$

$$\sum |a|^2 (b - c) \overset{a)}{=} \sum \frac{4}{(q+r)(\overline{q}+\overline{r})} \left(\frac{2rp}{r+p} - \frac{2pq}{p+q} \right) = \sum \frac{4qr}{(q+r)(q\overline{q}r+qr\overline{r})} \cdot \frac{2p^2\,(r-q)}{(r+p)(p+q)} =$$

$$= \frac{8pqr}{(p+q)(q+r)(r+p)} \sum \frac{p(r-q)}{q+r} = \frac{8pqr}{(p+q)^2(q+r)^2(r+p)^2} \sum p^3\,(r - q).$$

Because $\quad \sum p^3\,(r - q) = (p + q + r) \sum p^2\,(r - q),\quad$ we obtain:

$$\sum |a|^2 (b - c) = \frac{8pqr\,(p + q + r)}{(p + q)^2\,(q + r)^2\,(r + p)^2} \sum p^2\,(r - q). \qquad (1.3)$$

$$\sum \overline{a}\,(b - c) = \sum \frac{2\overline{q}\overline{r}}{\overline{q} + \overline{r}} \cdot \frac{2p^2\,(r - q)}{(p + q)\,(r + p)} = \frac{4}{(p + q)\,(q + r)\,(r + p)} \sum p^2\,(r - q) \qquad (1.4)$$

By dividing the equalities (1.3) şi (1.4), we obtain the conclusion.

c) From $h + 2o = a + b + c$, using **a**) and **b**), the conclusion follows.

Theorem 6. *We consider a triangle ABC, inscribed in the unit circle.*

a) $u, v, w \in \mathbb{C}$ *exist, with* $a = u^2$, $b = v^2$, $c = w^2$ *such that* $-uv, -uw, -vw$ *are the affixes of the midpoints of the arcs AB, AC and BC of the unit circle which don't contain the points C, B and A respectively.*

b) *The affix of the incenter I of the triangle ABC is* $i = -(uv + uw + vw)$.

Proof. **a**) Let M, N, P be the midpoints of the arcs BC, AC and AB of the unit circle which don't pass through A, B and C respectively.
The numbers $u_1, u_2, v_1, v_2, w_1, w_2 \in \mathbb{C}$ exist and are unique, such that

$$u_1^2 = u_2^2 = a,\ v_1^2 = v_2^2 = b,\ w_1^2 = w_2^2 = c.$$

From $\triangle\,BOM \equiv \triangle\,MOC,$ we find:

$\frac{m}{b} = \frac{c}{m},$ therefore

$$m^2 = bc = (v_k w_t)^2,\ \forall k, t \in \{1, 2\}.$$

Similarly, it follows:

$$n^2 = (u_s w_t)^2,\ \forall s, t \in \{1, 2\}.$$

We choose $u \in \{u_1, u_2\}, v \in \{v_1, v_2\},$
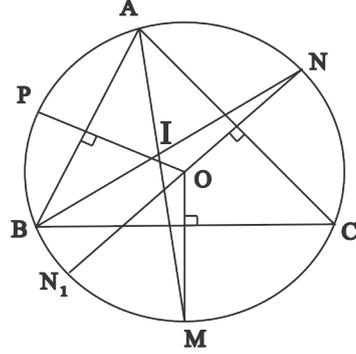$w \in \{w_1, w_2\}$ such that $m = -vw$ şi $n = -uw$.
Let $N_1\,(uw)$ be the diametrically opposite point of
N. From $\mu\left(\widehat{N_1 OP}\right) = A = \mu\left(\widehat{MOB}\right)$, it results
that $\frac{n_1}{p} = \frac{m}{b},$ therefore

$$p = \frac{bn_1}{m} = \frac{v^2 uw}{-vw} = -uv.$$

b) Because $\{I\} = AM \cap BN$, from **Theorem 4. c**) we deduce:

$i = \frac{am(b+n) - bn(a+m)}{am - bn} = \frac{-u^2 vw\left(v^2 - uw\right) + v^2 uw\left(u^2 - vw\right)}{-u^2 vw + v^2 uw} = -(uv + uw + vw).$

2. Problems

P1. *The incircle of the triangle ABC is the unit circle. We denote its center by I. Let D, E, F be the touchpoints of the incircle with BC, AC and AB respectively. We denote: $\{M\} = AI \cap DE$, $\{N\} = BI \cap EF$, $\{P\} = CI \cap FD$, $\{Q\} = AI \cap DF$. Prove that the following assertions are true:*

a) $IM \cdot IN \cdot IP = 1.$

b) I, E, C, Q *are concyclic.*

c) $BM = IM \cdot EC.$

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Solution. **a**) $ID = IE = IF = 1$ and using **Theorem 4. d**) it results:

$$a = \frac{2ef}{e+f}, \; b = \frac{2fd}{f+d}, \; c = \frac{2de}{d+e}.$$

We obtain: $\bar{a} = \frac{2}{e+f}$ $\bar{b} = \frac{2}{f+d}$ and $\bar{c} = \frac{2}{d+e}$

As $M \in DE$, we have:

$$m + de\overline{m} = d + e. \tag{2.1}$$

A, I, M are collinear, therefore

$$\frac{m}{a} = \frac{\overline{m}}{\overline{a}} \; \Leftrightarrow \; \overline{m} = m \cdot \frac{\overline{a}}{a} = \frac{m}{ef}.$$

By replacing in (2.1), it follows:

$$m + \frac{dm}{f} = d + e \; \Leftrightarrow \; m = \frac{(d+e) \cdot f}{d+f}. \tag{2.2}$$

As $Q \in DF$, we have:

$$q + fd\overline{q} = f + d. \tag{2.3}$$

A, I, Q are collinear, so $\overline{q} = q \cdot \frac{\overline{a}}{a} = \frac{q}{ef}.$

By replacing in (2.3), we deduce:

$$q = \frac{(d+f) \cdot e}{d+e}. \tag{2.4}$$

Similarly, we find:

$$n = \frac{(e+f) \cdot d}{e+d} \tag{2.5}$$

$$p = \frac{(f+d) \cdot e}{f+e} \tag{2.6}$$

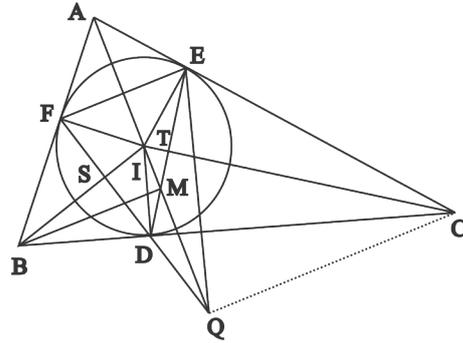By multiplying the equalities (2.2), (2.5) and (2.6), we obtain:

$mnp = def \; \Rightarrow \; IM \cdot IN \cdot IP = 1.$

  b) We denote: $\{S\} = BI \cap DF$, $\{T\} = CI \cap DE$.

From (2.2) it follows: $m = \frac{tf}{s}$, and using moduli, we have, $IM = \frac{IT}{IS}$.

From (2.4), it follows: $q = \frac{es}{t}$, and using moduli, we obtain $IQ = \frac{IS}{IT} = \frac{1}{IM}.$

Then, as $\frac{IQ}{IE} = \frac{IE}{IM}$ and $\widehat{QIE} \equiv \widehat{EIM}$, the triangles QIE and EIM are similar,

therefore $\mu\left(\widehat{IQE}\right) = \mu\left(\widehat{IEM}\right) = \frac{C}{2}$, which means that I, E, C, Q are concyclic.

c) $m = \frac{d+e}{2de} \cdot \frac{2def}{d+f} = \frac{be}{c}$, therefore $\frac{m}{b} = \frac{e}{c} \Leftrightarrow \triangle BIM \sim \triangle CIE$. We obtain:

$$BM = \frac{IM \cdot EC}{IE} = IM \cdot CE.$$

P2. *ABC is an arbitrary triangle and T is the intersection point of the tangents in B and C at its circumscribed circle. Let M be the midpoint of the segment BC and N the second intersection point of the line AT with the circle . Prove that $BM^2 = MN \cdot AM$.*

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Solution. We consider that the circumscribed circle of the triangle ABC is the unit circle.

Using **Theorem 4. d)** we obtain:

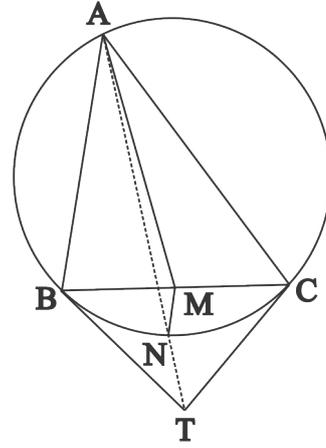$t = \frac{2bc}{b+c}$, thus $\bar{t} = \frac{2}{b+c}$.

Using **Theorem 3. a)** we deduce:

$$t + an\bar{t} = a + n \Leftrightarrow n = \frac{a - t}{a\bar{t} - 1}.$$

Replacing, we find: $n = \frac{ab+ac-2bc}{2a-b-c}$.

Then, $n - m = n - \frac{b+c}{2} = \frac{(b-c)^2}{4\left(a - \frac{b+c}{2}\right)}$.

Therfore $MN = \frac{BC^2}{4AM}$, and the conclusion follows.

P3. *The quadrilater $ABCD$ is inscribed in $\mathcal{C}(O,1)$, such that*

$$\mu\left(\widehat{AOB}\right) + \mu\left(\widehat{COD}\right) \neq \pi.$$

We denote by M and P the projections of the points A and C on the line BD. The points N and Q are the projections of B and D on the line AC.
Prove that $MNPQ$ is a parallelogram iff $ABCD$ is a rectangle.

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Solution. "$\Leftarrow$" The assertion is obvious. Moreover, $MNPQ$ is a rectangle.

"$\Rightarrow$" We choose the Cartesian coordinate system with the origin in the point O, and such that:

$a = \cos t_1 + i \cdot \sin t_1$, $b = \cos t_2 + i \cdot \sin t_2$, $c = \cos t_3 + i \cdot \sin t_3$, $d = \cos t_4 + i \cdot \sin t_4$, with $0 < t_1 < t_3 < t_4 < 2\pi$. We prove that $ac + bd \neq 0$.

Suppose that $ac + bd = 0$. It follows:

$$\cos(t_1 + t_3) + \cos(t_2 + t_4) + i\left(\sin(t_1 + t_3) + \sin(t_2 + t_4)\right) = 0 \Leftrightarrow$$

$$\Leftrightarrow 2\cos\frac{t_2 + t_4 - t_1 - t_3}{2}\left(\cos\frac{t_1 + t_2 + t_3 + t_4}{2} + i \cdot \sin\frac{t_1 + t_2 + t_3 + t_4}{2}\right) = 0 \Leftrightarrow$$

$\Leftrightarrow \cos \frac{t_2+t_4-t_1-t_3}{2} = 0 \Leftrightarrow (t_2-t_1)+(t_4-t_3) = \pi \Leftrightarrow \mu\left(\widehat{AOB}\right) + \mu\left(\widehat{COD}\right) = \pi$,

false. Therefore $ac+bd \neq 0$.

Using **Theorem 4. b**) we obtain:

$$m = \frac{b+d+a-bd\bar{a}}{2} = \frac{a+b+d}{2} - \frac{bd}{2a},$$

$$p = \frac{b+c+d-bd\bar{c}}{2} = \frac{b+c+d}{2} - \frac{bd}{2c},$$

$$n = \frac{a+b+c-ac\bar{b}}{2} = \frac{a+b+c}{2} - \frac{ac}{2b},$$

$$q = \frac{a+c+d-ac\bar{d}}{2} = \frac{a+c+d}{2} - \frac{ac}{2d}.$$

$m+p = n+q \Leftrightarrow \frac{a+b+d}{2} - \frac{bd}{2a} + \frac{b+c+d}{2} - \frac{bd}{2c} = \frac{a+b+c}{2} - \frac{ac}{2b} + \frac{a+c+d}{2} - \frac{ac}{2d}$.

It follows:

$$m+p = n+q \Leftrightarrow \frac{(b+d)(ac+bd)}{bd} = \frac{(a+c)(ac+bd)}{ac}.$$

As $ac+bd \neq 0$, we deduce: $\frac{b+d}{bd} = \frac{a+c}{ac} \Leftrightarrow ab(c-d) = cd(b-a)$.

Therefore $|a| \cdot |b| \cdot CD = |c| \cdot |d| \cdot AB$, so $AB = CD$.

Similarly, $\frac{b+d}{bd} = \frac{a+c}{ac} \Leftrightarrow bc(a-d) = ad(b-c)$, therefore $AD = BC$.

In the end, $ABCD$ is an inscribed parallelogram, namely $ABCD$ is a rectangle.

P4. *The point H is the orthocenter of the triangle ABC. The points D, E and F lie on the circumcircle of the triangle ABC such that $AD \parallel BE \parallel CF$. The points S, T and U are the respective reflections of D, E and F across the lines BC, CA and AB. Prove that the points S, T, U and H are concyclic.*

([2], **4, p. 7**)

Solution. We consider that the circumscribed circle of the triangle ABC is the unit circle, with the center O. Then, $h = a+b+c$.

$AD \parallel BE \Leftrightarrow \frac{a-d}{b-e} = \frac{\bar{a}-\bar{d}}{\bar{b}-\bar{e}} \Leftrightarrow \frac{a-d}{b-e} = \frac{d-a}{e-b} \cdot \frac{be}{ad} \Leftrightarrow be = ad$.

Similarly, $AD \parallel CF \Leftrightarrow ad = cf$.

$M(m)$ is the projection of the point D on BC.

Theorem 4. b) leads to

$$m = \frac{b+c+d-bc\bar{d}}{2}.$$

Because $s = 2m - d$, we obtain
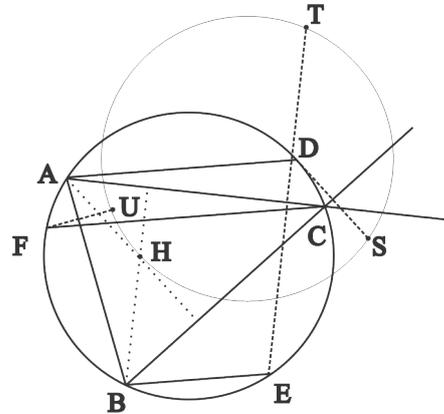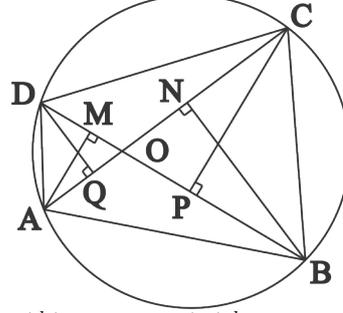
$$s = b+c-bc\bar{d}.$$

Similarly, we deduce:

$$t = a+c-ac\bar{e}$$

$$u = a+b-ab\bar{f}.$$

U, H, S, T are cyclic iff $\alpha = \frac{u-h}{t-h} \cdot \frac{t-s}{u-s} \in \mathbb{R}$.

We have $\alpha = \frac{-c-\frac{ab}{f}}{-b-\frac{ac}{e}} \cdot \frac{a-b-\frac{ac}{e}+\frac{bc}{d}}{a-c-\frac{ab}{f}+\frac{bc}{d}}$,

therefore $\alpha = \frac{(ab+cf)(ae-be)}{(ac+be)(af-fc)} = \frac{(ab+ad)(ae-ad)}{(ac+ad)(af-ad)} = \frac{(b+d)(e-d)}{(c+d)(f-d)}$.

From here,

$$\overline{\alpha} = \frac{\frac{1}{b} + \frac{1}{d}}{\frac{1}{c} + \frac{1}{d}} \cdot \frac{\frac{1}{e} - \frac{1}{d}}{\frac{1}{f} + \frac{1}{d}} = \frac{b+d}{bd} \cdot \frac{d-e}{de} \cdot \frac{cd}{c+d} \cdot \frac{df}{d-f} = \alpha \cdot \frac{cf}{be} = \alpha,$$

which leads to $\alpha \in \mathbb{R}$. Therefore, the points U, H, S, T are concyclic.

References

[1] *Muşuroia N. (coord.), Boroica Gh., Pop V., Heuberger D. (coord.), Bojor F., Matema-tică de excelenţă pentru concursuri, olimpiade şi centre de excelenţă, Clasa a X-a*, Ed. Paralela 45, Piteşti, 2013, 146-178

[2] *https://www.scribd.com/document/295267839/TJUSAMO-2013-2014-Complex-Numbers-in-Geometry*

[3] *https://www.yisun.io/teaching.html*

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