Some Elementary Infinite products of Trignometric Identities

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Abstract

In this paper I will be presenting some infinite products of trignometric identities

1 Introduction

The infinite product for the trigonometric sine function[1],[2] was introduced by the German mathematician leonhard Euler which is of the form

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right)$$

similarly for cosine function[2] we have

$$\cos x = \prod_{n=1}^{\infty}\left(1 - \frac{4x^2}{(2n-1)^2\pi^2}\right)$$

2 Infinite product for $\sin x + \cos x, \csc x + \sec x$, $\tan x + \cot x$

Theorem 1.

$$\sin x + \cos x = \frac{(4x+\pi)}{2\sqrt{2}}\prod_{n=1}^{\infty}\left(1 - \frac{(4x+\pi)^2}{16n^2\pi^2}\right) \tag{1}$$

Proof.

$$\begin{aligned}
\sin x + \cos x &= \sqrt{2} \sin\left(x + \frac{\pi}{4}\right) \\
&= \sqrt{2}\left(x + \frac{\pi}{4}\right) \frac{\sin\left(x + \frac{\pi}{4}\right)}{\left(x + \frac{\pi}{4}\right)} && \text{(Apply Euler's Sine product)} \\
&= \sqrt{2}\left(x + \frac{\pi}{4}\right) \prod_{n=1}^{\infty} \left(1 - \frac{(4x+\pi)^2}{16n^2\pi^2}\right) \\
&= \frac{(4x+\pi)}{2\sqrt{2}} \prod_{n=1}^{\infty} \left(1 - \frac{(4x+\pi)^2}{16n^2\pi^2}\right)
\end{aligned}$$

□

the $\sin x + \cos x$ can also be presented in terms of cosine product i.e

$$\sin x + \cos x = \sqrt{2} \prod_{n=1}^{\infty} \left(1 - \frac{(4x-\pi)^2}{(2n-1)^2 4\pi^2}\right) \tag{2}$$

if we substitute πx in Eq (1) we'll get

$$\sin \pi x + \cos \pi x = \pi \left(\frac{4x+1}{2\sqrt{2}}\right) \prod_{n=1}^{\infty} \left(1 - \frac{(4x+1)^2}{16n^2}\right) \tag{3}$$

similarly in Eq (2) we'll get

$$\sin \pi x + \cos \pi x = \sqrt{2} \prod_{n=1}^{\infty} \left(1 - \frac{(4x-1)^2}{4(2n-1)^2}\right) \tag{4}$$

when $x = \dfrac{1}{6}$ in Eq (3) and by rearranging it we can write the product as

$$\prod_{n=1}^{\infty} \left(1 - \frac{25}{144n^2}\right) = \frac{3\sqrt{2}\left(\sqrt{3}+1\right)}{5\pi}$$

similarly in Eq (4) we get

$$\prod_{n=1}^{\infty} \left(1 - \frac{1}{36(2n-1)^2}\right) = \frac{\left(\sqrt{3}+1\right)}{2\sqrt{2}}$$

Theorem 2.

$$\csc x + \sec x = \frac{(4x+\pi)}{2x\sqrt{2}} \prod_{n=1}^{\infty} \left(1 + \frac{64x^2 - (4x+\pi)^2}{16(\pi^2 n^2 - 4x^2)}\right) \tag{5}$$

Proof.

$$\begin{aligned}
\csc x + \sec x &= \frac{2x(\sin x + \cos x)}{x \sin 2x} \qquad \text{(Apply Theorem (1) and Sine product)} \\
&= \frac{(4x+\pi)}{x\sqrt{2}} \frac{\prod_{n=1}^{\infty} \left(1 - \frac{(4x+\pi)^2}{16n^2\pi^2}\right)}{\prod_{n=1}^{\infty} \left(1 - \frac{4x^2}{n^2\pi^2}\right)} \\
&= \frac{(4x+\pi)}{x\sqrt{2}} \prod_{n=1}^{\infty} \left(\frac{16\pi^2 n^2 - (4x+\pi)^2}{16(\pi^2 n^2 - 4x^2)}\right) \\
&= \frac{(4x+\pi)}{2x\sqrt{2}} \prod_{n=1}^{\infty} \left(1 + \frac{64x^2 - (4x+\pi)^2}{16(\pi^2 n^2 - 4x^2)}\right)
\end{aligned}$$

$\square$

if we substitute πx in Eq (5) we get

$$\csc(\pi x) + \sec(\pi x) = \frac{(4x+1)}{2\sqrt{2}} \prod_{n=1}^{\infty} \left(1 + \frac{48x^2 - 8x - 1}{16(n^2 - 4x^2)}\right) \tag{6}$$

for Eg when $x = \frac{1}{6}$ in Eq (6) and by rearranging we can write the product as

$$\prod_{n=1}^{\infty} \left(1 - \frac{9}{144n^2 - 16}\right) = \frac{2\sqrt{2}\left(\sqrt{3}+1\right)}{5\sqrt{3}}$$

Theorem 3.

$$\tan(x) + \cot(x) = \frac{1}{x} \prod_{n=1}^{\infty} \left(1 + \frac{4x^2}{n^2\pi^2 - 4x^2}\right) \tag{7}$$

Proof.

$$\begin{aligned}
\tan(x) + \cot(x) &= \frac{2}{\sin(2x)} && \text{(Apply Euler sine product)} \\
&= \frac{1}{x}\prod_{n=1}^{\infty}\left(\frac{n^2\pi^2}{n^2\pi^2 - 4x^2}\right) \\
&= \frac{1}{x}\prod_{n=1}^{\infty}\left(1 + \frac{4x^2}{n^2\pi^2 - 4x^2}\right)
\end{aligned}$$

$\square$

If we substitute πx in Eq (7) we'll get

$$\tan(\pi x) + \cot(\pi x) = \frac{1}{\pi x}\prod_{n=1}^{\infty}\left(1 + \frac{4x^2}{n^2 - 4x^2}\right) \tag{8}$$

for $x = \frac{1}{5}$ in Eq (8) we have the product

$$\prod_{n=1}^{\infty}\left(1 + \frac{4}{25n^2 - 4}\right) = \frac{\pi\left(6 - 2\sqrt{5}\right)}{5\sqrt{5 - 2\sqrt{5}}}$$

3 Infinite products of $\cos x - \cos y, \sin x - \sin y$

Theorem 4.

$$\cos(x) - \cos(y) = 2\sin^2\left(\frac{y}{2}\right)\left(1 - \frac{x^2}{y^2}\right)\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{(2n\pi + y)^2}\right)\left(1 - \frac{x^2}{(2n\pi - y)^2}\right) \tag{9}$$

Proof.

$$
\begin{aligned}
\cos(x)-\cos(y) &= 2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{y-x}{2}\right)\\
&= \left(\frac{y^2-x^2}{2}\right)\frac{\sin\left(\frac{x+y}{2}\right)}{\left(\frac{x+y}{2}\right)}\frac{\sin\left(\frac{y-x}{2}\right)}{\left(\frac{y-x}{2}\right)}\\
&= \left(\frac{y^2-x^2}{2}\right)\prod_{n=1}^{\infty}\left(1-\frac{(x+y)^2}{4n^2\pi^2}\right)\left(1-\frac{(y-x)^2}{4n^2\pi^2}\right)\\
&= \left(\frac{2}{y}\right)^2\sin^2\left(\frac{y}{2}\right)\left(\frac{y^2-x^2}{2}\right)\prod_{n=1}^{\infty}\frac{\left(4n^2\pi^2-(y+x)^2\right)\left(4n^2\pi^2-(y-x)^2\right)}{\left(4n^2\pi^2-(y)^2\right)^2}\\
&= 2\sin^2\left(\frac{y}{2}\right)\left(1-\frac{x^2}{y^2}\right)\prod_{n=1}^{\infty}\left(1-\frac{x^2}{(2n\pi+y)^2}\right)\left(1-\frac{x^2}{(2n\pi-y)^2}\right)
\end{aligned}
$$

□

when we substitute πx and πy in Eq (9) we get

$$
\cos(\pi x)-\cos(\pi y) = 2\sin^2\left(\frac{\pi y}{2}\right)\left(1-\frac{x^2}{y^2}\right)\prod_{n=1}^{\infty}\left(1-\frac{x^2}{(2n+y)^2}\right)\left(1-\frac{x^2}{(2n-y)^2}\right) \tag{10}
$$

when substituting $x=\dfrac{1}{4}$ and $y=\dfrac{1}{3}$ in Eq (10) and on rearranging we get

$$
\prod_{n=1}^{\infty}\left(1-\frac{9}{16(6n+1)^2}\right)\left(1-\frac{9}{16(6n-1)^2}\right)=\frac{16\left(\sqrt{2}-1\right)}{7}
$$

Theorem 5.

$$
\sin(x)-\sin(y)=(x-y)\prod_{n=1}^{\infty}\left(1-\frac{(x-y)^2}{4n^2\pi^2}\right)\left(1-\frac{(x+y)^2}{(2n-1)^2\pi^2}\right) \tag{11}
$$

Proof.

$$
\begin{aligned}
\sin(x)-\sin(y) &= 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)\\
&= 2\left(\frac{x-y}{2}\right)\frac{\sin\left(\frac{x-y}{2}\right)}{\left(\frac{x-y}{2}\right)}\cos\left(\frac{x+y}{2}\right) \qquad \text{(Apply Sine and Cosine product)}\\
&= (x-y)\prod_{n=1}^{\infty}\left(1-\frac{(x-y)^2}{4n^2\pi^2}\right)\left(1-\frac{(x+y)^2}{(2n-1)^2\pi^2}\right)
\end{aligned}
$$

$\square$

when we substitute πx and πy in Eq (11) we get

$$\sin(\pi x) - \sin(\pi y) = \pi\,(x-y)\prod_{n=1}^{\infty}\left(1-\frac{(x-y)^2}{4n^2}\right)\left(1-\frac{(x+y)^2}{(2n-1)^2}\right) \tag{12}$$

let us substitute $x=\frac{1}{3}$ and $y=\frac{1}{6}$ in Eq(12) and on rearranging we get

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{144n^2}\right)\left(1-\frac{1}{4\,(2n-1)^2}\right)=\frac{3\left(\sqrt{3}-1\right)}{\pi}$$

4 Conclusion

In this paper we came to see some proofs of infinite products for basic trignometric identities which are of the form $\sin x + \cos x$,$\tan x + \cot x$ and so , as metioned in the paper.

References

[1] Sumit Kumar Jha. On an elementary proof of euler's product expansion for the sine. 2019.

[2] Eric W Weisstein. Infinite product. *https://mathworld. wolfram. com/*, 2002.

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