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ROMANIAN MATHEMATICAL MAGAZINE

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SP.337 Let $x, y, z > 0$.

1) If $xy + yz + zx \leq 3(2\sqrt{3} - 3)$ then

$$\sqrt{\frac{xy + yz + zx}{3}} + 1 \leq \sqrt[3]{(x + 1)(y + 1)(z + 1)}.$$

2) If $xy + yz + zx > 3(2\sqrt{3} - 3)$ then

$$\sqrt{xy + yz + zx} + 1 < \sqrt{(x + 1)(y + 1)(z + 1)}.$$

Proposed by Florentin Vișescu-Romania

Solution by proposer

Let $xy + yz + zx = k > 0,\ z = \frac{k - xy}{x + y}$

Because $x, y, z > 0 \Rightarrow k - xy > 0$, or $x < \frac{k}{y}$.

Let $f(x) = (x + 1)(y + 1)(z + 1) = (x + 1)(y + 1)\left(\frac{k - xy}{x + y} + 1\right), f: \left(0, \frac{k}{y}\right) \to \mathbb{R}$.

$$f'(x) = (1 - y^2)\left(\frac{(x + y)^2 - (y^2 + k)}{(x + y)^2}\right) =$$

$$= \frac{(1 - y)^2\left(x + y - \sqrt{y^2 + k}\right)\left(x + y + \sqrt{y^2 + k}\right)}{(x + y)^2}.$$

1. If $y = 1 \Rightarrow f'(x) = 0 \Rightarrow f(x) = \text{ct} = 2(x + 1)\left(\frac{k - x}{x + 1} + 1\right) = 2(k + 1)$.

2. If $y < 1 \Rightarrow 1 - y^2 > 0$.

$$f'(x) = 0 \Rightarrow x + y - \sqrt{y^2 + k} = 0 \Rightarrow x = \sqrt{y^2 + k} - y > 0.$$

We show that:

$$\sqrt{y^2 + k} - y < \frac{k}{y} \Leftrightarrow \sqrt{y^2 + k} < \frac{y^2 + k}{y} \Leftrightarrow y < \sqrt{y^2 + k} \Leftrightarrow y^2 < y^2 + k \Leftrightarrow$$

$0 < k$, true.

x	0	$\sqrt{y^2+k}-y$	$\frac{k}{y}$
$f'(x)$	$-----------$	0	$+++++++++$
$f(x)$	$\searrow\searrow\searrow\searrow\searrow\searrow$	1	$\nearrow\nearrow\nearrow\nearrow\nearrow\nearrow$

$$f\left(\sqrt{y^2+k}-k\right)=(y+1)\left(\sqrt{y^2+k}-y+1\right)^2.(1)$$

Then $f(x)\geq(y+1)\left(\sqrt{y^2+k}-y+1\right)^2,\forall\, x\in\left(0,\frac{k}{y}\right),y<1.$

Let $g(y)=(y+1)\left(\sqrt{y^2+k}-y+1\right)^2, g:(0,1)\to\mathbb{R}.$

$$g'(y)=\left(\sqrt{y^2+k}-y+1\right)\cdot\frac{3y^2-3y\sqrt{y^2+k}-\sqrt{y^2+k}+k+2y}{\sqrt{y^2+k}}.$$

$$g'(y)=0\Rightarrow 3y^2-3y\sqrt{y^2+k}-\sqrt{y^2+k}+k+2y=0\Leftrightarrow$$

$$3y^2+2y+k=\sqrt{y^2+k}(3y+1)\Leftrightarrow 6y^3+3y^2-3y^2k-2ky+k^2-k=0\Leftrightarrow$$

$$(2y+1-k)(3y^2-k)=0\Rightarrow y_1=\frac{k-1}{2}\text{sau}y_2=\sqrt{\frac{k}{3}}.$$

a)If $k\leq 1\Rightarrow y_1=\frac{k-1}{2}\leq 0$ and $0<y_2=\sqrt{\frac{k}{3}}<1.$

y	0	$\sqrt{k/3}$	1
$g'(y)$	$-----------$	0	$+++++++++$
$g(y)$	$\searrow\searrow\searrow\searrow$	2	$\nearrow\nearrow\nearrow\nearrow$

$$g\left(\sqrt{\frac{k}{3}}\right)=\left(\sqrt{\frac{k}{3}}+1\right)\cdot\left(\sqrt{\frac{k}{3}+k}-\sqrt{\frac{k}{3}}+1\right)^2=\left(\sqrt{\frac{k}{3}}+1\right)^3.(2)$$

Then $f(x)\geq\left(\sqrt{\frac{k}{3}}+1\right)^3,\forall\, x\in\left(0;\frac{k}{y}\right),y<1,k\leq 1.$

b) If $k \in (1;3] \Rightarrow 0 < \frac{k-1}{2} \le \sqrt{\frac{k}{3}} \le 1.$

We show that:

$$\frac{k-1}{2} \le \sqrt{\frac{k}{3}} \Leftrightarrow 3k^2 - 6k + 3 \le 4k \Leftrightarrow 3k^2 - 10k + 3 \le 0 \Leftrightarrow k \in \left[\frac{1}{3};3\right], \text{true}.$$

y	0 $\quad \frac{k-1}{2}$ $\quad \sqrt{\frac{k}{3}}$ $\quad 1$
$g'(y)$	$+ + + + 0 - - - - - - - - 0 + + + +$
$g(y)$	↗ ↗ ↗ ↗ 3 ↘ ↘ ↘ ↘ 2 ↗ ↗ ↗

$$\lim_{\substack{y\to 0\\ y>0}} g(y) = \left(\sqrt{k}+1\right)^2 . (3)$$

$$g\left(\sqrt{\frac{k}{3}}\right) = \left(\sqrt{\frac{k}{3}}+1\right)^3 . (2)$$

Compare $\left(\sqrt{k}+1\right)^2$ with $\left(\sqrt{\frac{k}{3}}+1\right)^3$.

$$\left(\sqrt{k}+1\right)^2 < \left(\sqrt{\frac{k}{3}}+1\right)^3 \Leftrightarrow k + 2\sqrt{k} + 1 < \frac{k}{3}\sqrt{\frac{k}{3}} + 3\frac{k}{3} + 3\sqrt{\frac{k}{z}} + 1 \Leftrightarrow k > 3\left(2\sqrt{3}-3\right).$$

Then $f(x) \ge \left(\sqrt{\frac{k}{3}}+1\right)^3, \forall\, x \in \left(0,\frac{k}{y}\right), y < 1, k \in \left(1;3\left(2\sqrt{3}-3\right)\right]$

and $f(x) > \left(\sqrt{k}+1\right)^2, \forall\, x \in \left(0;\frac{k}{y}\right), y < 1, k \in \left(3\left(2\sqrt{3}-3\right);3\right].$

c) If $k > 3 \Rightarrow 1 < \sqrt{\frac{k}{3}} < \frac{k-1}{2}.$

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y	0	1
$g'(y)$	$+ + + + + + + + + + + + + + + + + +$	
$g(y)$	$3\ \nearrow\ \nearrow\ \nearrow\ \nearrow\ \nearrow\ \nearrow\ \nearrow$	

$$\lim_{\substack{y\to 0\\ y>0}} g(y)=\left(\sqrt{k}+1\right)^2. (3)$$

Then $f(x)>\left(\sqrt{k}+1\right)^2, \forall\, x\in\left(0;\frac{k}{y}\right), y<1, k\in(3,\infty)$.

3)If $y>1\Rightarrow 1-y^2<0$.

x	0	$\sqrt{y^2+k}-y$	$\frac{k}{y}$
$f'(x)$	$+ + + + + + +$	0	$- - - - - - - -$
$f(x)$	$\nearrow\ \nearrow\ \nearrow\ \nearrow\ \nearrow\ \nearrow$	4	$\searrow\ \searrow\ \searrow\ \searrow\ \searrow$

$$\lim_{\substack{x\to 0\\ x>0}} f(x)=(y+1)\left(\frac{k}{y}+1\right). (4), \lim_{x\to\frac{k}{y}} f(x)=\left(\frac{k}{y}+1\right)(y+1). (4')$$

Then $f(x)>(y+1)\left(\frac{k}{y}+1\right), \forall\, x\in\left(0;\frac{k}{y}\right), y>1$.

Let $h(y)=(y+1)\left(\frac{k}{y}+1\right), h:(1;\infty)\to\mathbb{R}. h'(y)=\frac{k}{y}+1-\frac{k}{y^2}(y+1)$

$$=\frac{k}{y}+1-\frac{k}{y}-\frac{k}{y^2}=\frac{y^2-k}{y^2}. h'(y)=0\Rightarrow y^2=k\Rightarrow y=\sqrt{k}.$$

a) If $k\in(0;1]\ \Rightarrow y=\sqrt{k}\le 1$.

y	0	∞
$h'(y)$	$+ + + + + + + + + + + + + + + + + +$	
$h(y)$	$5\ \nearrow\ \nearrow\ \nearrow\ \nearrow\ \nearrow\ \nearrow\ \nearrow$	

$$\lim_{\substack{y\to 1\\ y>1}} h(y)=2(k+1). (5)$$

Then $f(x) > 2(k+1), \forall\, x \in \left(0; \frac{k}{y}\right), y > 1, k \in (0,1]$.

b) If $k > 1 \Rightarrow y = \sqrt{k} > 1$

y	1 $\qquad$ $\sqrt{k}$ $\qquad$ ∞
$h'(y)$	$- - - - - - - - 0 + + + + + + + + +$
$h(y)$	↘ ↘ ↘ ↘ ↘ ↘ 6 ↗ ↗ ↗ ↗ ↗ ↗ ↗

$$h(\sqrt{k}) = (\sqrt{k}+1)^2. (6)$$

Then $f(x) > (\sqrt{k+1})^2, \forall\, x \in \left(0; \frac{k}{y}\right), y > 1, k > 1$.

I showed that:

$$\text{for } k \in (0,1] \left(\sqrt{\frac{k}{3}}+1\right)^3 < 2(k+1) \Leftrightarrow \frac{k}{3}\sqrt{\frac{k}{3}} + 3\cdot\frac{k}{3} + 3\sqrt{\frac{k}{3}} + 1 < 2k+2 \Leftrightarrow$$

$$\frac{k}{3}\sqrt{\frac{k}{3}} - 3\frac{k}{3} + 3\sqrt{\frac{k}{3}} - 1 < 0 \Leftrightarrow \left(\sqrt{\frac{k}{3}} - 1\right)^3 < 0 \Leftrightarrow k < 3, \text{true}.$$

In conclusion:

For $xy+yz+zx \le 3(2\sqrt{3}-3) \Rightarrow \sqrt{\frac{xy+yz+zx}{3}} + 1 \le \sqrt[3]{(x+1)(y+1)(z+1)}$.

For $xy+yz+zx > 3(2\sqrt{3}-3) \Rightarrow \sqrt{xy+yz+zx} + 1 < \sqrt{(x+1)(y+1)(z+1)}$.

Note by editor:

Many thanks to Florică Anastase-Romania for typed solution.