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# QUASI-EXACT DIFFERENTIAL EQUATION

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**Abstract**
This article examines a specific variant of exact differential equation whose form is a matrix product of exact partial derivatives and a modifying vector.

## 1. Introduction

It is widely perceived (see e.g. [1]) that if $P(x,y)$ and $Q(x,y)$ are two bivariable function such that:

$$\frac{\partial[P(x,y)]}{\partial y}=\frac{\partial[Q(x,y)]}{\partial x} \tag{1}$$

Then the exact first-ordered ordinary differential equation:

$$P(x,y)\,dx+Q(x,y)dy=0 \tag{2}$$

Has solutions as:

$$\begin{cases} U(x,y)=const \\ \left[\begin{array}{l} U(x,y)=\displaystyle\int_{x_0}^{x}P(x,y_0)\,dx+\int_{y_0}^{y}Q(x,y)\,dy+const \\ U(x,y)=\displaystyle\int_{x_0}^{x}P(x,y)\,dx+\int_{y_0}^{y}Q(x_0,y)\,dy+const \end{array}\right. \end{cases} \tag{3}$$

Of which, $U(x,y)$ is the potential function of (1) where exactness of the above differential equation is determined by the criteria that:

$$\begin{cases} \dfrac{\partial[U(x,y)]}{\partial x}=P(x,y) \quad \dfrac{\partial[U(x,y)]}{\partial y}=Q(x,y) \\ \dfrac{\partial^2[U(x,y)]}{\partial x\partial y}=\dfrac{\partial[P(x,y)]}{\partial y}=\dfrac{\partial[Q(x,y)]}{\partial x} \end{cases} \tag{4}$$

Accordingly, we define the quasi-exact differential equation (QDE) as:

$$P(x,y)\,dx+R(x,y)Q(x,y)dy=0 \tag{5}$$

An alternate form of QDE is $vw=0$ where $v=[P(x,y)\,dx \quad Q(x,y)dy]$ are exact partial derivatives, $w=[1 \quad R(x,y)]'$ is the modifying vector, and $R(x,y)$ is the modifier.
We shall solve (5) where $R(x,y)$ is (i) a constant in Section 2; and a variable function in Section 3, and Section 4, respectively. Following, Section 5 provides discussion.

**2. Constant quasi-exact differential equation**

In the case $R(x,y)$ is a constant $r$ ($r\neq 0$ and $r\neq 1$)[1], the constant QDE becomes:

$$P(x,y)\,dx + rQ(x,y)dy = 0 \tag{6}$$

We shall transform (6) to the exact form by multiplying both sides with an integrating factor $S(x,y)$ which is not a constant. Equation (6) becomes:

$$P(x,y)S(x,y)\,dx + rQ(x,y)S(x,y)dy = 0 \tag{7}$$

It is supposed to find $S(x,y)$ such that:

$$\frac{\partial[P(x,y)S(x,y)]}{\partial y} = \frac{\partial[rQ(x,y)S(x,y)]}{\partial x}$$
$$\Leftrightarrow S\frac{\partial P}{\partial y} + P\frac{\partial S}{\partial y} = r\left(S\frac{\partial Q}{\partial x} + Q\frac{\partial S}{\partial x}\right)$$
$$\Leftrightarrow S\left(\frac{\partial P}{\partial y} - r\frac{\partial Q}{\partial x}\right) + P\frac{\partial S}{\partial y} - rQ\frac{\partial S}{\partial x} = 0$$

With $U(x,y)$ as determined under (3) and (4), we could rewrite the above as:

$$S\left(\frac{\partial^2 U}{\partial x\partial y} - r\frac{\partial^2 U}{\partial x\partial y}\right) + \frac{\partial U\partial S}{\partial x\partial y} - r\frac{\partial U\partial S}{\partial x\partial y} = 0$$
$$\Leftrightarrow S(1-r)\frac{\partial^2 U}{\partial x\partial y} + (1-r)\frac{\partial U\partial S}{\partial x\partial y}$$
$$\Leftrightarrow S\partial^2 U + \partial U\partial S = 0$$

Since $S(x,y)\neq const$, we could divide both sides of the above by $\partial S^2$ and consequently obtain a second-ordered ordinary differential equation:

$$S\frac{\partial^2 U}{\partial S^2} + \frac{\partial U}{\partial S} = 0 \Leftrightarrow S\frac{d^2 U}{dS^2} + \frac{dU}{dS} = 0 \tag{8}$$

In equation (8), replacing $T = dU/dS$ gives:

$$S\frac{dT}{dS} + T = 0 \Leftrightarrow \frac{dT}{T} = -\frac{dS}{S} \Leftrightarrow \ln|T| = -\ln|S| + c \Leftrightarrow T = \frac{k}{S}$$

Henceforth, we obtain:

$$\frac{dU}{dS} = \frac{k}{S} \Leftrightarrow dU = k\frac{dS}{S} \Leftrightarrow U = k\ln|S| + l \Leftrightarrow S = ae^{bU}$$

The finding that $S = ae^{bU}$ gives solutions of (7) are $W(x,y) = const$, of which:

$$\begin{aligned} W(x,y) &= \int_{x_0}^{x} P(x,y_0)S(x,y_0)\,dx + r\int_{y_0}^{y} Q(x,y)S(x,y)\,dy + const \\ &= \int_{x_0}^{x} ae^{bU(x,y_0)}\left.\frac{\partial U(x,y)}{\partial x}\right|_{y=y_0} dx + r\int_{y_0}^{y} ae^{bU(x,y)}\frac{\partial U(x,y)}{\partial y}dy + const \end{aligned} \tag{9}$$

Or:

[1] If $r = 0$, solutions are $P(x,y) = 0$ or $x = const$; if $r = 1$, the QDE becomes an exact differential equation.

$$W(x,y)=\int_{x_0}^{x}P(x,y)S(x,y)\,dx+r\int_{y_0}^{y}Q(x_0,y)S(x_0,y)\,dy+const$$
$$=\int_{x_0}^{x}ae^{bU(x,y)}\frac{\partial U(x,y)}{\partial x}dx+r\int_{y_0}^{y}ae^{bU(x_0,y)}\frac{\partial U(x,y)}{\partial y}\bigg|_{x=x_0}dy+const \quad (10)$$

***Example 1***

Solve the following differential equation:

$$(ye^x+e^y)\,dx=(xe^y+e^x)dy \quad (11)$$

Equation (11) is quasi-exact where $P(x,y)=ye^x+e^y$, $Q(x,y)=xe^y+e^x$, and $r=-1$. Besides, formula (3) gives $U(x,y)=xe^y+ye^x$.
It is found that the integrating factor is:

$$S(x,y)=ae^{bU(x,y)}=ae^{b(xe^y+ye^x)}$$

Thus, solution of Example 1 is $W(x,y)=const$, where:

$$W(x,y)=\int_{x_0}^{x}ae^{b(xe^{y_0}+y_0e^x)}(y_0e^x+e^{y_0})\,dx-\int_{y_0}^{y}ae^{b(xe^y+ye^x)}(xe^y+e^x)\,dy+const$$
$$W(x,y)=\frac{a}{b}e^{b(xe^{y_0}+y_0e^x)}\Big|_{x_0}^{x}-\frac{a}{b}e^{b(xe^y+ye^x)}\Big|_{y_0}^{y}+const$$
$$W(x,y)=\frac{2a}{b}e^{b(xe^{y_0}+y_0e^x)}-\frac{a}{b}e^{b(xe^y+ye^x)}+const$$

Simplifying $a=b$, then a solution of (11) is $W(x,y)=const$, where:

$$W(x,y)=2e^{b(xe^{y_0}+y_0e^x)}-e^{b(xe^y+ye^x)}+const$$

An alternate expression is:

$$W(x,y)=e^{b(xe^y+ye^x)}-2e^{b(x_0e^y+ye^{x_0})}+const$$

## 3. Univariable quasi-exact differential equation

We shall solve the QDE (5) when $R(x,y)\neq const$. A unique case is:

$$\frac{\partial(RQ)}{\partial x}=\frac{\partial Q}{\partial x}\Leftrightarrow Q\frac{\partial R}{\partial x}+R\frac{\partial Q}{\partial x}=\frac{\partial Q}{\partial x}\Leftrightarrow Q\frac{\partial R}{\partial x}=(1-R)\frac{\partial Q}{\partial x}\Leftrightarrow\frac{\partial R}{1-R}=\frac{\partial Q}{Q}$$

The above case results in a separable differential equation:

$$\frac{dR}{1-R}=\frac{dQ}{Q}\Leftrightarrow \ln|1-R|=\ln|Q|+c\Leftrightarrow 1-R=kQ\Leftrightarrow R=1-kQ$$

Of which, $k=const$. In this case, (5) becomes exact whose solution is under the form of (3).
For $R(x,y)\neq 1-kQ(x,y)$, it is supposed to find the integrating factor $S(x,y)\neq const$ such that:

$$\begin{cases}P(x,y)S(x,y)\,dx+R(x,y)Q(x,y)S(x,y)dy=0\\ \dfrac{\partial[P(x,y)S(x,y)]}{\partial y}=\dfrac{\partial[R(x,y)Q(x,y)S(x,y)]}{\partial x}\end{cases} \quad (12)$$

Finding $S(x,y)$:

$$S\frac{\partial P}{\partial y}+P\frac{\partial S}{\partial y}=QS\frac{\partial R}{\partial x}+RS\frac{\partial Q}{\partial x}+RQ\frac{\partial S}{\partial x}$$
$$\Leftrightarrow S\frac{\partial^2 U}{\partial x\partial y}+\frac{\partial U\partial S}{\partial x\partial y}=RS\frac{\partial^2 U}{\partial x\partial y}+R\frac{\partial U\partial S}{\partial x\partial y}+S\frac{\partial R\partial U}{\partial x\partial y} \quad (13)$$

This includes a specific case that $\partial R/\partial x=0\Leftrightarrow R(x,y)=R(y)$ or $\partial R/\partial x=0\Leftrightarrow R(x,y)=R(x)$. In this case, the process of finding $S(x,y)$ turns equivalent to Section 2.
Henceforth, the QDE (5) is comprehensively solvable if $R(x,y)$ is a univariable function. Without loss of generality, we assume $R(x,y)=R(y)$ and (13) becomes:

$$S\frac{\partial^2 U}{\partial x\partial y}+\frac{\partial U\partial S}{\partial x\partial y}=RS\frac{\partial^2 U}{\partial x\partial y}+R\frac{\partial U\partial S}{\partial x\partial y}$$
$$\Leftrightarrow S\partial^2 U+\partial U\partial S=RS\partial^2 U+R\partial U\partial S$$
$$\Leftrightarrow S(1-R)\partial^2 U+(1-R)\partial U\partial S=0$$

Since $R\neq const$ and $S\neq const$, we could transform the above equation as:

$$S\frac{\partial^2 U}{\partial S^2}+\frac{\partial U}{\partial S}=0\Leftrightarrow S\frac{d^2 U}{dS^2}+\frac{dU}{dS}=0\Leftrightarrow U=k\ln|S|+l\Leftrightarrow S=ae^{bU}$$

Thus, solutions of (5) are $W(x,y)=const$, where:

$$W(x,y)=\int_{x_0}^{x}P(x,y_0)S(x,y_0)\,dx+\int_{y_0}^{y}R(y)Q(x,y)S(x,y)\,dy+const$$
$$=\int_{x_0}^{x}ae^{bU(x,y_0)}\left.\frac{\partial U(x,y)}{\partial x}\right|_{y=y_0}dx+\int_{y_0}^{y}aR(y)e^{bU(x,y)}\frac{\partial U(x,y)}{\partial y}dy+const \quad (14)$$

Or:

$$W(x,y)=\int_{x_0}^{x}P(x,y)S(x,y)\,dx+\int_{y_0}^{y}R(y)Q(x_0,y)S(x_0,y)\,dy+const$$
$$=\int_{x_0}^{x}ae^{bU(x,y)}\frac{\partial U(x,y)}{\partial x}dx+\int_{y_0}^{y}aR(y)e^{bU(x_0,y)}\left.\frac{\partial U(x,y)}{\partial y}\right|_{x=x_0}dy+const \quad (15)$$

***Example 2***

Solve the QDE:

$$\left(\ln y+\frac{y}{x}\right)dx+(x+y\ln x)\,dy=0 \quad (16)$$

Equation (16) is quasi-exact where:

$$\begin{cases}P(x,y)=\ln y+\dfrac{y}{x}\\ Q(x,y)=\ln x+\dfrac{x}{y}\\ R(x,y)=y\end{cases}$$

The exact differential formula gives:

$$\begin{cases} U(x,y) = x\ln y + y\ln x \\ S(x,y) = ae^{bU(x,y)} = axye^{b(x+y)} \end{cases}$$

Thus, solution of (16) is $W(x,y) = const$, where:

$$W(x,y) = \int_{x_0}^{x} axy_0\left(\ln y_0 + \frac{y_0}{x}\right)e^{b(x+y_0)}\,dx + \int_{y_0}^{y} axy(x + y\ln x)e^{b(x+y)}\,dy + const$$

Or:

$$W(x,y) = \int_{x_0}^{x} axy\left(\ln y + \frac{y}{x}\right)e^{b(x+y)}\,dx + \int_{y_0}^{y} ax_0y(x_0 + y\ln x_0)e^{b(x_0+y)}\,dy + const$$

## 4. Bivariable generalized quasi-exact differential equation

This Section considers equation (13) in a generalized case that $\partial R/\partial x \neq 0$ and $\partial R/\partial y \neq 0$. Accordingly, we obtain:

$$S\partial^2 U + \partial U\partial S = RS\partial^2 U + R\partial U\partial S + S\partial R\partial S$$

$$\Leftrightarrow S\frac{\partial^2 U}{\partial S^2} + \frac{\partial U}{\partial S} = RS\frac{\partial^2 U}{\partial S^2} + R\frac{\partial U}{\partial S} + S\frac{\partial R}{\partial S}$$

$$\Leftrightarrow S(1-R)\frac{\partial^2 U}{\partial S^2} + (1-R)\frac{\partial U}{\partial S} - S\frac{\partial R}{\partial S} = 0$$

Thus, we obtain a second-ordered partial differential equation:

$$\frac{\partial^2 U}{\partial S^2} + \left(\frac{1}{S}\right)\frac{\partial U}{\partial S} + \left(\frac{1}{R-1}\right)\frac{\partial R}{\partial S} = 0 \quad (17)$$

Due to coherent characteristics, equation (17) is extremely hard to solve in a generalized manner, especially when $S(x,y)$ is not constant. On the other hand, a feasible direction is assuming that solutions of (17) are $U = U(S,R)$ and somehow transforming (17) to a linear second-ordered partial differential equation (see e.g. [2], [3], [4], and [5]).

## 5. Discussion

We have examined the quasi-exact differential equation which is obtained by modifying the exact differential form. We have shown that the QDE is solvable when the modifier is either constant or univariable. To solve the QDE comprehensively, we suggest three potential pathways for further researches which include (i) upgrading the order, e.g. second-ordered QDE; (ii) generalizing the number of variables, e.g. QDE in vector space; and (iii) generalizing the solution for the integrating factor as indicated by a complicated partial differential equation, i.e. (17).

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Solve the following partial differential equation:

$$\begin{cases} \dfrac{\partial u(x;t)}{\partial t} = k\dfrac{\partial^2 u(x;t)}{\partial x^2} \\ u(x;0) = 0 \\ u(0;t) = h \end{cases}$$

Of which, $x > 0$, $t > 0$, $k > 0$ and $h$ are constant.

The problem is a heat equation on $(0;+\infty)$ with homogeneous initial conditions and constant boundary conditions, and therefore has solutions expressed as:

$$u(x;t) = \int_0^t \frac{x}{\sqrt{4k\pi(t-s)^3}} exp\left[-\frac{x^2}{4k(t-s)}\right] h(s)\,ds$$

$$= \frac{2h}{\sqrt{\pi}} \int_0^t \frac{x}{4\sqrt{k(t-s)^3}} exp\left[-\left(\frac{x}{2\sqrt{k(t-s)}}\right)^2\right] ds$$

Transforming $y = x/\left[2\sqrt{k(t-s)}\right]$, $dy$ and boundary values are determined as:

$$\begin{cases} \dfrac{dy}{ds} = \dfrac{x}{4\sqrt{k(t-s)^3}} \\ \lim\limits_{s\to t^-} y = +\infty \\ y(s=0) = \dfrac{x}{2\sqrt{kt}} \end{cases}$$

Thus, we could rewrite the solution:

$$u(x;t)=\frac{2h}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{kt}}}^{+\infty}e^{-y^2}\,dy=h\left(\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{kt}}}^{0}e^{-y^2}\,dy+\frac{2}{\sqrt{\pi}}\int_{0}^{+\infty}e^{-y^2}\,dy\right)$$

$$=h\left(1-\frac{2}{\sqrt{\pi}}\int_{0}^{\frac{x}{2\sqrt{kt}}}e^{-y^2}\,dy\right)=h\left[1-erf\left(\frac{x}{2\sqrt{kt}}\right)\right]=h\times erfc\left(\frac{x}{2\sqrt{kt}}\right)$$

In the above calculation, we employ the Poisson's integral (given $a>0$) that:

$$I=\int_0^{+\infty}e^{-ax^2}\,dx=\int_0^{+\infty}e^{-ay^2}\,dy>0\Rightarrow I^2=\iint_0^{+\infty}e^{-a(x^2+y^2)}\,dx\,dy$$

Replacing $x=r\cos\varphi$ and $y=r\sin\varphi$, we get $\{r;\varphi\}\in(0;\pi/2)\times(0;+\infty)$ and:

$$\frac{\partial(x;y)}{\partial(r;\varphi)}=\begin{vmatrix}\frac{\partial x}{\partial r}=\cos\varphi & \frac{\partial x}{\partial\varphi}=-r\sin\varphi\\ \frac{\partial y}{\partial r}=\sin\varphi & \frac{\partial y}{\partial\varphi}=r\cos\varphi\end{vmatrix}=r$$

Henceforth, the square of Poisson's integral is:

$$I^2=\int_0^{\frac{\pi}{2}}\int_0^{+\infty}e^{-ar^2}\,r\,dr\,d\varphi=\int_0^{\frac{\pi}{2}}d\varphi\int_0^{+\infty}re^{-ar^2}\,dr=\frac{\pi}{2}\int_0^{+\infty}re^{-ar^2}\,dr=\frac{\pi}{4a}\int_0^{+\infty}e^{-ar^2}\,d(ar^2)$$

$$=\frac{\pi}{4a}\int_0^{+\infty}e^{-z}\,dz=\frac{\pi}{4a}e^{-z}\Big|_{+\infty}^{0}=\frac{\pi}{4a}$$

Therefore:

$$I=\frac{\sqrt{\pi}}{2\sqrt{a}}$$

*Saigon, 11 août 2020*

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*As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality.*

**Albert Einstein**

*God exists since mathematics is consistent, and the Devil exists since we cannot prove it.*

**Andre Weil**