Integral Involving Square of Logarithm Function

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Given the integral,

$$I = \int_0^{\frac{\pi}{2}} ln^2\left(\frac{e^{-x^2}}{\cos(x)}(1 + cos4x)\right) dx$$

We can rewrite the integral as,

$$I = \int_0^{\frac{\pi}{2}} \left[ln\left(e^{-x^2}\right) + ln\left(\frac{1 + cos4x}{\cos(x)}\right)\right]^2 dx$$

$$= \int_0^{\frac{\pi}{2}} ln^2\left(e^{-x^2}\right)dx + 2\int_0^{\frac{\pi}{2}} ln\left(e^{-x^2}\right)ln\left(\frac{1 + cos4x}{\cos(x)}\right) dx + \int_0^{\frac{\pi}{2}} ln^2\left(\frac{1 + cos4x}{\cos(x)}\right) dx$$

$$= \quad I_1 \quad + \quad I_2 \quad + \quad I_3$$

Let us now evaluate each integral.

To evaluate I_1:

$$I_1 = \int_0^{\frac{\pi}{2}} ln^2\left(e^{-x^2}\right)dx = \int_0^{\frac{\pi}{2}} \left[ln\left(e^{-x^2}\right)\right]^2 dx = \int_0^{\frac{\pi}{2}} x^4 dx = \frac{\pi^5}{160}$$

To evaluate I_2:

$$I_2 = 2\int_0^{\frac{\pi}{2}} ln\left(e^{-x^2}\right) ln\left(\frac{1+cos4x}{\cos(x)}\right) dx = -2\int_0^{\frac{\pi}{2}} x^2[\ln(1+cos4x) - \ln(cosx)]dx$$

$$= -2\int_0^{\frac{\pi}{2}} x^2 \ln(1+cos4x)\, dx + 2\int_0^{\frac{\pi}{2}} x^2 \ln(cosx)\, dx = I_{21} + I_{22}$$

To evaluate I_{21}:

There is a singularity at $= \frac{\pi}{4}$. Therefore integral can be rewritten as,

$$I_{21} = -2\int_0^{\frac{\pi}{4}} x^2 \ln(1+cos4x)\, dx - 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x^2 \ln(1+cos4x)\, dx = I_{211} + I_{212}$$

Consider I_{211}:

We can make the following substitutions first.

Let $u = 2x$ $\qquad x : 0\ to\ \frac{\pi}{4}$

$du = 2dx$ $\qquad u : 0\ to\ \frac{\pi}{2}$

$$\therefore\ I_{211} = -\frac{1}{4}\int_0^{\frac{\pi}{2}} u^2 \ln(1+cos2u)\, du = -\frac{1}{4}\int_0^{\frac{\pi}{2}} u^2 \ln(2cos^2u)\, du$$

$$= -\frac{1}{4}\int_0^{\frac{\pi}{2}} u^2[ln2 + ln(cos^2u)]du = -\frac{ln2}{4}\int_0^{\frac{\pi}{2}} u^2 du - \frac{1}{4}\int_0^{\frac{\pi}{2}} u^2 \ln(cos^2u)\, du$$

$$I_{211} = -\frac{\pi^3}{96} ln2 - \frac{1}{2}\int_0^{\frac{\pi}{2}} u^2 \ln(\cos(u))\, du$$

From the result (iii) we obtain,

$$I_{211} = -\frac{\pi^3}{96} ln2 - \frac{1}{2}\left(-\frac{\pi^3}{24} ln2 - \frac{\pi}{4}\zeta(3)\right) = \frac{\pi^3}{96} ln2 + \frac{\pi}{8}\zeta(3)$$

Consider I_{212}:

$$I_{212} = -2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x^2 \ln(1 + cos4x)\, dx$$

Let $u = 2x$ $\qquad x : \frac{\pi}{4}\ to\ \frac{\pi}{2}$

$du = 2dx$ $\qquad u : \frac{\pi}{2}\ to\ \pi$

$$\therefore\ I_{212} = -\frac{1}{4}\int_{\frac{\pi}{2}}^{\pi} u^2 \ln(1 + cos2u)\, du = -\frac{1}{4}\int_{\frac{\pi}{2}}^{\pi} u^2 \ln(2cos^2u)\, du$$

$$= -\frac{1}{4}\int_{\frac{\pi}{2}}^{\pi} u^2[ln2 + \ln(cos^2u)]du = -\frac{ln2}{4}\int_{\frac{\pi}{2}}^{\pi} u^2 du - \frac{1}{4}\int_{\frac{\pi}{2}}^{\pi} u^2 \ln(cos^2(u))\, du$$

$$= -\frac{7\pi^3}{96} ln2 - \frac{1}{4}\int_{\frac{\pi}{2}}^{\pi} u^2 \ln(cos^2(u))\, du$$

Now let us make another substitution.

Let $t = \pi - u \qquad\qquad u : \frac{\pi}{2} \ to\ \pi$

$dt = -du \qquad\qquad t : \frac{\pi}{2} \ to\ 0$

$$\therefore \ I_{212} = -\frac{7\pi^3}{96} ln2 - \frac{1}{4}\int_0^{\frac{\pi}{2}} (\pi - t)^2 ln(cos^2 t)dt$$

Where

$$\int_0^{\frac{\pi}{2}} (\pi - t)^2 ln(cos^2 t)dt = \int_0^{\frac{\pi}{2}} (\pi^2 - 2\pi t + t^2) ln(cos^2 t)dt$$

$$= 2\int_0^{\frac{\pi}{2}} \pi^2 \ln(\cos(t))\, dt - 4\pi \int_0^{\frac{\pi}{2}} tln(\cos(t))dt + 2\int_0^{\frac{\pi}{2}} t^2 \ln(\cos(t))\, dt$$

By using the results (i), (ii) and (iii) we get,

$$\int_0^{\frac{\pi}{2}} (\pi - t)^2 ln(cos^2 t)dt = 2\pi^2\left[-\frac{\pi}{2} ln2\right] - 4\pi\left[-\frac{\pi^2}{8} ln2 - \frac{7}{16}\zeta(3)\right] + 2\left[-\frac{\pi^3}{24} ln2 - \frac{\pi}{4}\zeta(3)\right]$$

$$= -\frac{7\pi^3}{12} ln2 + \frac{5\pi}{4}\zeta(3)$$

$$\therefore \quad I_{212} = -\frac{7\pi^3}{96} ln2 - \frac{1}{4}\int_0^{\frac{\pi}{2}} (\pi - t)^2 ln(cos^2 t)dt$$

$$= -\frac{7\pi^3}{96} ln2 + \frac{7\pi^3}{48} ln2 - \frac{5\pi}{16}\zeta(3) = \frac{7\pi^3}{96} ln2 - \frac{5\pi}{16}\zeta(3)$$

$$i.e.\ I_{21} = -2\int_0^{\frac{\pi}{2}} x^2 \ln(1 + cos4x)\, dx = I_{211} + I_{212}$$

$$= \frac{\pi^3}{12} ln2 - \frac{3\pi}{16}\zeta(3)$$

Now we have,

$$I_{22} = 2\int_0^{\frac{\pi}{2}} x^2 \ln(cosx)\, dx$$

By using the result (iii) we get,

$$I_{22} = -\frac{\pi^3}{12} ln2 - \frac{\pi}{2}\zeta(3)$$

$$\therefore \quad I_2 = 2\int_0^{\frac{\pi}{2}} ln\left(e^{-x^2}\right) ln\left(\frac{1 + cos4x}{\cos(x)}\right) dx = I_{21} + I_{22} = -\frac{11\pi}{16}\zeta(3)$$

Now consider

$$I_3 = \int_0^{\frac{\pi}{2}} ln^2\left(\frac{1 + cos4x}{cosx}\right) dx$$

We can rewrite it as.

$$I_3 = \int_0^{\frac{\pi}{2}} [\ln(1 + cos4x) - ln(cosx)]^2 dx = \int_0^{\frac{\pi}{2}} ln^2(1 + cos4x)dx -$$

$$2\int_0^{\frac{\pi}{2}} \ln(1 + cos4x)\ln(cosx)\,dx + \int_0^{\frac{\pi}{2}} ln^2(cosx)dx = I_{31} + I_{32} + I_{33}$$

Consider I_{31}:

$$I_{31} = \int_0^{\frac{\pi}{2}} ln^2(1 + cos4x)dx = 2\int_0^{\frac{\pi}{4}} ln^2(1 + cos4x)dx$$

Let $u = 2x$ $\quad$ $x : 0\ to\ \frac{\pi}{4}$

$du = 2dx$ $\quad$ $u: 0\ to\ \frac{\pi}{2}$

$$I_{31} = \int_0^{\frac{\pi}{2}} ln^2(1 + cos2u)du = \int_0^{\frac{\pi}{2}} ln^2(2cos^2u)du$$

$$= \int_0^{\frac{\pi}{2}} [ln2 + \ln(cos^2u)]^2 du = \int_0^{\frac{\pi}{2}} ln^2 2du + 2\int_0^{\frac{\pi}{2}} ln2\ln(cos^2u)\,du +$$

$$\int_0^{\frac{\pi}{2}} ln^2(cos^2u)du = I_{311} + I_{312} + I_{313}$$

$$I_{311} = \int_0^{\frac{\pi}{2}} ln^2 2du = \frac{\pi}{2} ln^2 2$$

$$I_{312} = 2\int_0^{\frac{\pi}{2}} ln2 \ln(cos^2 u)\, du = 4\int_0^{\frac{\pi}{2}} ln2 \ln(cosu)\, du$$

Using the result (i) we get,

$$I_{312} = -2\pi ln^2 2$$

Now

$$I_{313} = \int_0^{\frac{\pi}{2}} ln^2(cos^2 u)du = 4\int_0^{\frac{\pi}{2}} ln^2(cosu)du$$

Using the result (iv) we get,

$$I_{313} = 4\left[\frac{\pi^3}{24} + \frac{\pi}{2} ln^2 2\right] = \frac{\pi^3}{6} + 2\pi ln^2 2$$

$$\therefore \quad I_{31} = \int_0^{\frac{\pi}{2}} ln^2(1 + cos2u)du = \frac{\pi^3}{6} + \frac{\pi}{2} ln^2 2$$

Now consider

$$I_{32} = -2\int_0^{\frac{\pi}{2}} \ln(1 + cos4x) \ln(cosx)\, dx = -2\int_0^{\frac{\pi}{2}} \ln(2cos^2 2x) \ln(cosx)\, dx$$

$$I_{32} = -2\int_0^{\frac{\pi}{2}} [\ln 2 + \ln(cos^2 2x)] \ln(cosx)\, dx = -2\int_0^{\frac{\pi}{2}} ln2 \ln(cosx)\, dx$$

$$-2\int_0^{\frac{\pi}{2}} \ln(cos^2 2x) \ln(cosx)\, dx = I_{321} + I_{322}$$

$$I_{321} = -2\int_0^{\frac{\pi}{2}} ln2 \ln(cosx)\, dx = \pi ln^2 2$$

$$I_{322} = -2\int_0^{\frac{\pi}{2}} \ln(cos^2 2x) \ln(cosx)\, dx = -4\int_0^{\frac{\pi}{2}} \ln(cos2x) \ln(cosx)\, dx$$

By using the result (v) we get,

$$I_{322} = -4\left[\frac{\pi}{2} ln^2 2 - \frac{\pi^3}{96}\right] = \frac{\pi^3}{24} - 2\pi ln^2 2$$

$$\therefore \;\; I_{32} = -2\int_0^{\frac{\pi}{2}} \ln(1 + cos4x) \ln(cosx)\, dx = \frac{\pi^3}{24} - \pi ln^2 2$$

Now consider our last integral,

$$I_{33} = \int_0^{\frac{\pi}{2}} ln^2(\cos x) dx = \frac{\pi^3}{24} + \frac{\pi}{2} ln^2 2 \quad (using\ the\ result\ (iv))$$

$$I_3 = \int_0^{\frac{\pi}{2}} ln^2\left(\frac{1+cos4x}{cosx}\right)dx = I_{31} + I_{32} + I_{33} = \frac{\pi^3}{4}$$

Now we can combine all the integrals to get the final solution.

$$i.e. \qquad I = \int_0^{\frac{\pi}{2}} ln^2\left(\frac{e^{-x^2}}{\cos(x)}(1+cos4x)\right)dx = I_1 + I_2 + I_3$$

$$\boxed{\int_0^{\frac{\pi}{2}} ln^2\left(\frac{e^{-x^2}}{\cos(x)}(1+cos4x)\right)dx = \frac{\pi^5}{160} - \frac{11\pi}{16}\zeta(3) + \frac{\pi^3}{4}}$$

RESULTS :

$$\int_0^{\frac{\pi}{2}} \ln(\cos(x)\,dx = -\frac{\pi}{2}ln2 \qquad (i)$$

$$\int_0^{\frac{\pi}{2}} xln(cosx)dx = -\frac{\pi^2}{8}ln2 - \frac{7}{16}\zeta(3) \qquad (ii)$$

$$\int_0^{\frac{\pi}{2}} x^2ln(cosx)dx = -\frac{\pi^3}{24}ln2 - \frac{\pi}{4}\zeta(3) \qquad (iii)$$

$$\int_0^{\frac{\pi}{2}} ln^2(cosx)dx = \frac{\pi^3}{24} + \frac{\pi}{2} ln^2 2 \qquad (iv)$$

$$\int_0^{\frac{\pi}{2}} \ln(cosax)\ln(cosbx)\,dx = \frac{\pi}{2} ln^2 2 - \frac{\pi^3}{48}\frac{gcd^2(a,b)}{48ab} \qquad (v)$$

$$if \ \frac{a+b}{\gcd(a,b)} \ is\ odd$$

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