Property of $\psi(m,a)$

If,

$$\psi(m,a)=\int_0^\infty \frac{e^{-x^2}x^{m-1}}{(1+x^2)^a}dx$$

where $m>0$ and $a\in\mathbb{R}$, then prove that,

$$\Big(\frac{e\pi erfc(1)}{2}\Big)^2=\frac{1}{2}\sum_{k=0}^{\infty}(-1)^k B\Big(\frac{1}{2}+k,\frac{1}{2}+k\Big)\psi(4k+2,k+1)$$

where $B(x,y)$ is the Beta function.
Proof: Consider the following product,

$$\psi(m,a)\psi(n,a)=\int_0^\infty \frac{e^{-x^2}x^{m-1}}{(1+x^2)^a}dx\int_0^\infty \frac{e^{-y^2}y^{n-1}}{(1+y^2)^a}dy=\int_0^\infty\int_0^\infty \frac{e^{-(x^2+y^2)}x^{m-1}y^{n-1}}{(1+x^2)^a(1+y^2)^a}dxdy$$

Transforming cartesian coordinates into polar coordinates, we obtain,

$$\psi(m,a)\psi(n,a)=\int_0^{\frac{\pi}{2}}\int_0^\infty \frac{e^{-r^2}(rcos(\theta))^{m-1}(rsin(\theta))^{n-1}}{(1+r^2+r^4sin^2(\theta)cos^2(\theta))^a}rdrd\theta$$

Thus,

$$\psi(m,a)\psi(n,a)=\int_0^{\frac{\pi}{2}}cos^{m-1}(\theta)sin^{n-1}(\theta)\int_0^\infty \frac{e^{-r^2}r^{m+n-1}}{(1+r^2+r^4sin^2(\theta)cos^2(\theta))^a}drd\theta$$

Taking $(1+r^2)^a$ common from the denominator and then expanding the denominator using binomial theorem, we get,

$$\psi(m,a)\psi(n,a)=\int_0^{\frac{\pi}{2}}cos^{m-1}(\theta)sin^{n-1}(\theta)\int_0^\infty \frac{e^{-r^2}r^{m+n-1}}{(1+r^2)^a}\sum_{k=0}^{\infty}\frac{(-a)_k}{k!}\Big(\frac{r^4sin^2(\theta)cos^2(\theta)}{1+r^2}\Big)^k drd\theta$$

Now, inverting the order of integration and summation and using the known identity $(-a)_k=(-1)^k a^{(k)}$, we get,

$$\psi(m,a)\psi(n,a)=\sum_{k=0}^{\infty}\frac{(-1)^k a^{(k)}}{k!}\int_0^\infty \frac{e^{-r^2}r^{4k+m+n-1}}{(1+r^2)^{k+a}}dr\int_0^{\frac{\pi}{2}}cos^{m+2k-1}(\theta)sin^{n+2k-1}(\theta)d\theta$$

It is well known that,

$$\int_0^{\frac{\pi}{2}}cos^p(\theta)sin^q(\theta)d\theta=\frac{B(\frac{p+1}{2},\frac{q+1}{2})}{2}$$

Thus, using the above identity and with some manipulations,

$$2\psi(m,a)\psi(n,a) = \sum_{k=0}^{\infty} \frac{(-1)^k a^{(k)}}{k!} \psi(4k+m+n, k+a) B\Big(\frac{m}{2}+k, \frac{n}{2}+k\Big)$$

Now substituting $m = n = a = 1$, we obtain,

$$\psi^2(1,1) = \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k B\Big(\frac{1}{2}+k, \frac{1}{2}+k\Big)\psi(4k+2, k+1)$$

It can be easily shown that,

$$\psi(1,1) = \frac{e\pi erfc(1)}{2}$$

by letting

$$I(n) = \int_0^{\infty} \frac{e^{-nx^2}}{1+x^2} dx$$

and then by forming a diferential equation,

$$I(n) - I^{'}(n) = \frac{\sqrt{\pi}}{2\sqrt{n}}$$

solving it we obtain,

$$I(n) = \frac{1}{2}\pi e^n erfc(\sqrt{n})$$

Put $n = 1$ to obtain the desired result.

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