R M M

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ABOUT AN INEQUALITY BY VASILE MIRCEA POPA-II

Proposed by Marin Chirciu – Romania

1) If $x,y,z>0, x+y+z=\frac{3}{2}$ then:

$$\frac{x}{1+y}+\frac{y}{1+z}+\frac{z}{1+x}\geq 1$$

Proposed by Vasile Mircea Popa – Romania

Solution *Using Bergström's inequality, we obtain:*

$\frac{x}{1+x}+\frac{y}{1+y}+\frac{z}{1+z}=\frac{x^2}{x+xy}+\frac{y^2}{y+yz}+\frac{z^2}{z+zx}\geq\frac{(x+y+z)^2}{x+y+z+xy+yz+zx}=\frac{\frac{9}{4}}{\sum xy+\frac{3}{2}}\geq 1$, *the last inequality is equivalent with* $\frac{9}{4}\geq 3(xy+yz+zx)\Leftrightarrow(x+y+z)^2\geq 3(xy+yz+zx)\Leftrightarrow$

$\Leftrightarrow(x-y)^2+(y-z)^2+(z-x)^2\geq 0$, *obviously, with equality if and only if*

$x=y=z=\frac{1}{2}$.

Remark. *The inequality can be developed.*

2) If $\mathrm{x},\mathrm{y},\mathrm{z}>0, \mathrm{x}+\mathrm{y}+\mathrm{z}=\frac{3}{2}$ and $\mathrm{n}\geq 0$, then:

$$\frac{x}{n+y}+\frac{y}{n+z}+\frac{z}{n+x}\geq\frac{3}{2n+1}$$

Proposed by Marin Chirciu – Romania

Solution *Using Berström's inequality:*

$\frac{x}{n+y}+\frac{y}{n+z}+\frac{z}{n+x}=\frac{x^2}{nx+xy}+\frac{y^2}{ny+yz}+\frac{z^2}{nz+zx}\geq\frac{(x+y+z)^2}{n(x+y+z)+xy+yz+zx}=\frac{\frac{9}{4}}{\sum xy+\frac{3n}{2}}\geq\frac{3}{2n+1}$ *where the last inequality is equivalent with* $\frac{9}{4}\geq 3(xy+yz+zx)\Leftrightarrow$

$\Leftrightarrow(x+y+z)^2\geq 3(xy+yz+zx)\Leftrightarrow(x-y)^2+(y-z)^2+(z-x)^2\geq 0$, *obviously, with equality if and only if* $x=y=z=\frac{1}{2}$.

Note: For $n=1$ we obtain problem VIII. 23, from RMM-24, Spring Edition 2020, Vasile Mircea Popa.

3) If $x, y, z > 0, x + y + z = \frac{3}{2}$ and $n \geq 0$, then:

$$\frac{x}{1 + ny} + \frac{y}{1 + nz} + \frac{z}{1 + nx} \geq \frac{3}{n + 2}$$

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Solution *Using Bergström we obtain:*

$$\frac{x}{1 + ny} + \frac{y}{1 + nz} + \frac{z}{1 + nx} = \frac{x^2}{x + nxy} + \frac{y^2}{y + nyz} + \frac{z^2}{z + nzx} \geq$$

$$\geq \frac{(x + y + z)^2}{x + y + z + n(xy + yz + zx)} = \frac{\frac{9}{4}}{n \sum xy + \frac{3}{2}} \geq \frac{3}{n + 2}$$

where the last inequality is equivalent with $\frac{9n}{4} \geq 3n(xy + yz + zx)$

For $n = 0$ *is obvious, and for* $n > 0$ *is equivalent with* $(x + y + z)^2 \geq 3(xy + yz + zx)$

$\Leftrightarrow (x - y)^2 + (y - z)^2 + (z - x)^2 \geq 0$, *obviously, with equality if and only if*

$$x = y = z = \frac{1}{2}.$$

Note. For $n = 1$, we obtain problem VIII.23, from RMM-24, Spring Edition 2020, Vasile Mircea Popa.

Reference:

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