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ABOUT AN INEQUALITY BY RAHIM SHAHBAZOV-VIII

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1) In ΔABC the following relationship holds:

$$\frac{h_a^2}{bc}+\frac{h_b^2}{ca}+\frac{h_c^2}{ab}\le\frac{9}{4}$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan

Solution. **Lemma. 2) In ΔABC the following relationship holds:**

$$\frac{h_a^2}{bc}+\frac{h_b^2}{ca}+\frac{h_c^2}{ab}=\frac{s^2+r^2+4Rr}{4R^2}$$

Proof. Using identity $h_a=\frac{2F}{a}$, we get:

$$\sum\frac{h_a^2}{bc}=\sum\frac{\left(\frac{2F}{a}\right)^2}{bc}=\frac{4F^2}{abc}\sum\frac{1}{a}=\frac{4r^2s^2}{4Rrs}\frac{ab+bc+ca}{abc}=\frac{rs}{R}\frac{ab+bc+ca}{4Rrs}$$
$$=\frac{s^2+r^2+4Rr}{4R^2}$$

Let's get back to the main problem.

Using Lemma and $s^2\le 4R^2+4Rr+3r^2(Gerretsen)$, we get:

$$LHS=\sum\frac{h_a^2}{bc}=\frac{s^2+r^2+4Rr}{4R^2}\le\frac{4R^2+4Rr+3r^2+r^2+4Rr}{4R^2}=$$
$$=\frac{4(R+r)^2}{4R^2}=\left(\frac{R+r}{R}\right)^2\overset{(1)}{\le}\frac{9}{4}=RHS,(1)\Leftrightarrow\left(\frac{R+r}{R}\right)^2\le\frac{9}{4}\Leftrightarrow$$
$$\frac{R+r}{R}\le\frac{3}{2}\Leftrightarrow R\ge 2r(Euler)$$

Equality holds if and only if triangle is equilateral.Remark. Inequality can be much stronger.

3) In ΔABC the following relationship holds:

$$\frac{h_a^2}{bc}+\frac{h_b^2}{ca}+\frac{h_c^2}{ab}\le\left(1+\frac{r}{R}\right)^2$$

Proposed by Marin Chirciu-Romania

Solution. See up these solution. Equality holds if and only if triangle is equilateral.

Remark. Let's find an reverse inequality.

5) In ΔABC the following relationship holds:

$$\frac{h_a^2}{bc}+\frac{h_b^2}{ca}+\frac{h_c^2}{ab}\ge\left(\frac{3r}{R}\right)^2$$

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Solution. Lemma. 6) In ΔABC the following relationship holds:

$$\frac{h_a^2}{bc}+\frac{h_b^2}{ca}+\frac{h_c^2}{ab}=\frac{s^2+r^2+4Rr}{4R^2}$$

Proof. Using identity $h_a=\frac{2F}{a}$, we get:

$$\sum\frac{h_a^2}{bc}=\sum\frac{\left(\frac{2F}{a}\right)^2}{bc}=\frac{4F^2}{abc}\sum\frac{1}{a}=\frac{4r^2s^2}{4Rrs}\frac{ab+bc+ca}{abc}=\frac{rs}{R}\frac{ab+bc+ca}{4Rrs}$$
$$=\frac{s^2+r^2+4Rr}{4R^2}$$

Let's get back to the main problem.

Using Lemma and $s^2\ge 16Rr-5r^2(Gerretsen)$, we get:

$$LHS=\sum\frac{h_a^2}{bc}=\frac{s^2+r^2+4Rr}{4R^2}\ge\frac{16Rr-5r^2+r^2+4Rr}{4R^2}=\frac{4r(5R-r)}{4R^2}=$$
$$=\frac{r(5R-r)}{R^2}\overset{Euler}{\ge}\frac{r(5\cdot 2r-r)}{R^2}=\frac{9r^2}{R^2}=\left(\frac{3r}{R}\right)^2=RHD.$$

Equality holds if and only if triangle is equilateral.

7) In ΔABC the following <=relationship holds:

$$\left(\frac{3r}{R}\right)^2\le\frac{h_a^2}{bc}+\frac{h_b^2}{ca}+\frac{h_c^2}{ab}\le\left(1+\frac{r}{R}\right)^2$$

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Solution. **Lemma. 8) In ΔABC the following relationship holds:**

$$\frac{h_a^2}{bc}+\frac{h_b^2}{ca}+\frac{h_c^2}{ab}=\frac{s^2+r^2+4Rr}{4R^2}$$

Proof. Using identity $h_a=\frac{2F}{a}$, we get:

$$\sum\frac{h_a^2}{bc}=\sum\frac{\left(\frac{2F}{a}\right)^2}{bc}=\frac{4F^2}{abc}\sum\frac{1}{a}=\frac{4r^2s^2}{4Rrs}\frac{ab+bc+ca}{abc}=\frac{rs}{R}\frac{ab+bc+ca}{4Rrs}$$
$$=\frac{s^2+r^2+4Rr}{4R^2}$$

Let's get back to the main problem.

For RHS, using Lemma and $s^2\le 4R^2+4Rr+3r^2(Gerretsen)$, we have:

$$E=\sum\frac{h_a^2}{bc}=\frac{s^2+r^2+4Rr}{4R^2}\le\frac{4R^2+4Rr+3r^2+r^2+4Rr}{4R^2}=\frac{4(R+r)^2}{4R^2}=\left(\frac{R+r}{R}\right)^2$$

$=RHS$. Equality holds if and only if triangle is equilateral.

For LHS, using Lemma and $s^2\ge 16Rr-5r^2(Gerretsen)$, we have:

$$LHS=\sum\frac{h_a^2}{bc}=\frac{s^2+r^2+4Rr}{4R^2}\ge\frac{16Rr-5r^2+r^2+4Rr}{4R^2}=\frac{4r(5R-r)}{4R^2}\overset{Euler}{\ge}$$
$$\ge\frac{r(5\cdot 2r-r)}{R^2}=\frac{9r^2}{R^2}=\left(\frac{3r}{R}\right)^2=RHS$$

Equality holds if and only if triangle is equilateral.

Remark. Replacing h_a with r_a , it follows:

9) In ΔABC the following relationship holds:

$$\frac{9}{4}\le\frac{r_a^2}{bc}+\frac{r_b^2}{ca}+\frac{r_c^2}{ab}\le\frac{9}{4}\left(\frac{R}{2r}\right)^4$$

Proposed by Marin Chirciu-Romania

Solution. Lemma. 10) In ΔABC the following relationship holds:

$$\frac{r_a^2}{bc}+\frac{r_b^2}{ca}+\frac{r_c^2}{ab}=\frac{8R^2+2Rr-s^2}{2Rr}$$

Proof. Using identity $r_a=\frac{F}{s-a}$, we get:

$$\sum\frac{r_a^2}{bc}=\sum\frac{\left(\frac{F}{s-a}\right)^2}{bc}=F^2\sum\frac{1}{bc(s-a)^2}=r^2s^2\frac{8R^2+2Rr-s^2}{2Rr^3s^2}=\frac{8R^2+2Rr-s^2}{2Rr}$$

true from $\sum\frac{1}{bc(s-a)^2}=\frac{8R^2+2Rr-s^2}{2Rr^3s^2}$ which follows from

$$\sum\frac{1}{bc(s-a)^2}=\frac{\sum a(s-b)^2(s-c)^2}{abc\prod(s-a)^2}=\frac{2sr^2(8R^2+2Rr-s^2)}{4Rrs\cdot sr^2\cdot sr^2}=\frac{8R^2+2Rr-s^2}{2Rr^3s^2}$$

$$\sum a(s-b)^2(s-c)^2=2sr^2(8R^2+2Rr-s^2)$$

Let's get back to the main problem.

For RHS, using Lemma and $s^2\geq 16Rr-5r^2(Gerretsen)$, we have:

$$E=\sum\frac{r_a^2}{bc}=\frac{8R^2+2Rr-s^2}{2Rr}\leq\frac{8R^2+2Rr-16Rr+5r^2}{2Rr}=$$

$$=\frac{8R^2-14Rr+5r^2}{2Rr}\overset{(1)}{\leq}\frac{9}{4}\left(\frac{R}{2r}\right)^4=RHS,$$

$$(1)\Leftrightarrow\frac{8R^2-14Rr+5r^2}{2Rr}\leq\frac{9}{4}\left(\frac{R}{2r}\right)^4\Leftrightarrow 32r^3(8R^2-14Rr+5r^2)\leq 9R^5\Leftrightarrow$$

$$9R^5-256R^2r^3=448Rr^4-160r^5\geq 0\Leftrightarrow$$

$(R-2r)(9R^4+18R^3r+36R^2r^2-184Rr^3+80r^4)\geq 0$, true from $R\geq 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

For LHS, using Lemma and $s^2,=4R^2+4Rr+3r^2(Gerretsen)$, we have:

$$LHS=\sum\frac{r_a^2}{bc}=\frac{8R^2+2Rr-s^2}{2Rr}\geq\frac{8R^2+2Rr-4R^2-4Rr-3r^2}{2Rr}\geq$$

$$\geq\frac{4R^2-2Rr-3r^2}{2Rr}\overset{(2)}{\geq}\frac{9}{4}=RHS$$

$$(2)\Leftrightarrow\frac{4R^2-2Rr-3r^2}{2Rr}\geq\frac{9}{4}\Leftrightarrow 8R^2-13Rr-6r^2\geq 0\Leftrightarrow(R-2r)(8R+3r)\geq 0$$

true from $R\geq 2r(Euler)$. Equality holds if and only if triangle is equilateral.

11) In ΔABC the following relationship holds:

$$\frac{h_a^2}{bc}+\frac{h_b^2}{ca}+\frac{h_c^2}{ab}\leq\frac{r_a^2}{bc}+\frac{r_b^2}{ca}+\frac{r_c^2}{ab}$$

Proposed by Marin Chirciu-Romania

Solution. **Using Lemmas, we have:**

$$\frac{h_a^2}{bc}+\frac{h_b^2}{ca}+\frac{h_c^2}{ab}=\frac{s^2+r^2+4Rr}{4R^2},\frac{r_a^2}{bc}+\frac{r_b^2}{ca}+\frac{r_c^2}{ab}=\frac{8R^2+2Rr-s^2}{2Rr}$$

$$\frac{s^2+r^2+4Rr}{4R^2}\leq\frac{8R^2+2Rr-s^2}{2Rr}\Leftrightarrow s^2(2R+r)\leq 16R^3+4R^2r-4Rr^2-r^3,$$

which follows from $s^2\leq 4R^2+4Rr+3r^2(Gerretsen)$. **Remains to prove that:**

$$(4R^2+4Rr+3r^2)(2R+r)\leq 16R^3+4R^2r-4Rr^2-r^3\Leftrightarrow$$

$$(R-2r)(2R+r)^2\geq 0 \text{ true from } R\geq 2r(Euler).$$

Equality holds if and only if triangle is equilateral.

12) In ΔABC the following relationship holds:

$$\left(\frac{3r}{R}\right)^2\leq\frac{h_a^2}{bc}+\frac{h_b^2}{ca}+\frac{h_c^2}{ab}\leq\frac{r_a^2}{bc}+\frac{r_b^2}{ca}+\frac{r_c^2}{ab}\leq\frac{8R^2-14Rr+5r^2}{2Rr}$$

Proposed by Marin Chirciu-Romania

Solution. **See up these inequalities. Equality if and only if triangle is equilateral.**

Remark. Replacing h_a with m_a, it follows:

13) In ΔABC the following relationship holds:

$$\frac{9}{4}\leq\frac{m_a^2}{bc}+\frac{m_b^2}{ca}+\frac{m_c^2}{ab}\leq\frac{9R}{8r}$$

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Solution. **Lemma. 14) In ΔABC the following relationship holds:**

$$\frac{m_a^2}{bc}+\frac{m_b^2}{ca}+\frac{m_c^2}{ab}=\frac{s^2+5r^2+2Rr}{8Rr}$$

Proof. Using identity: $m_a^2=\frac{2b^2+2c^2-a^2}{4}$, **we get:**

$$\sum\frac{m_a^2}{bc}=\sum\frac{\frac{2b^2+2c^2-a^2}{4}}{bc}=\frac{1}{4}\sum\frac{2b^2+2c^2-a^2}{bc}=\frac{1}{4}\frac{\sum a(2b^2+2c^2-a^2)}{abc}=$$

$$=\frac{1}{4}\frac{2\sum a(b^2+c^2)-\sum a^3}{abc}=\frac{2\cdot 2s(s^2+r^2-2Rr)-2s(s^2-3r^2-6Rr)}{4abc}=$$

$=\frac{2s(s^2+5r^2+2Rr)}{4\cdot 4Rrs}=\frac{s^2+5r^2+2Rr}{8Rr}$, which follows from

$$\sum a(b^2+c^2)=2s(s^2+r^2-2Rr),\sum a^3=2s(s^2-3r^2-6Rr).$$

Let's get back to the main problem.

For RHS, using Lemma and $s^2\le 4R^2+4Rr+3r^2(Gerretsen)$, we get:

$$E=\sum\frac{m_a^2}{bc}=\frac{s^2+5r^2+2Rr}{8Rr}\le\frac{4R^2+4Rr+3r^2+5r^2+2Rr}{8Rr}=$$

$$=\frac{2R^2+3Rr+4r^2}{4Rr}\overset{(1)}{\le}\frac{9R}{8r}=RHS$$

$$(1)\Leftrightarrow\frac{2R^2+3Rr+4r^2}{4Rr}\le\frac{9R}{8r}\Leftrightarrow 2(2R^2+3Rr+4r^2)\le 9R^2\Leftrightarrow$$

$5R^2-6Rr-8r^2\ge 0\Leftrightarrow(R-2r)(5R+4r)\ge 0$ true from $R\ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

For LHS, using Lemma and $s^2\ge 16Rr-5r^2(Gerretsen)$, we get:

$$LHS=\sum\frac{m_a^2}{bc}=\frac{s^2+5r^2+2Rr}{8Rr}\ge\frac{16Rr-5r^2+5r^2+2Rr}{8Rr}=\frac{9}{4}=RHS.$$

Equality holds if and only if triangle is equilateral.

15) In ΔABC the following relationship holds:

$$\frac{h_a^2}{bc}+\frac{h_b^2}{ca}+\frac{h_c^2}{ab}\le\frac{m_a^2}{bc}+\frac{m_b^2}{ca}+\frac{m_c^2}{ab}\le\frac{r_a^2}{bc}+\frac{r_b^2}{ca}+\frac{r_c^2}{ab}$$

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Solution. The first inequality follows from $h_a\le m_a$ and the second from up these lemmas:

$$\frac{m_a^2}{bc}+\frac{m_b^2}{ca}+\frac{m_c^2}{ab}=\frac{s^2+5r^2+2Rr}{8Rr},\frac{r_a^2}{bc}+\frac{r_b^2}{ca}+\frac{r_c^2}{ab}=\frac{8R^2+2Rr-s^2}{2Rr}$$

$\frac{s^2+5r^2+2Rr}{8Rr}\le\frac{8R^2+2Rr-s^2}{2Rr}\Leftrightarrow 5s^2\le 32R^2+6R-5r^2$, which follows from

$s^2\le 4R^2+4Rr+3r^2(Gerretsen)$. Remains to prove that:

$$5(4R^2+4Rr+3r^2)\le 32R^2+6Rr-5r^2\Leftrightarrow 6R^2-7Rr-10r^2\ge 0$$

$\Leftrightarrow(R-2r)(6R+5r)\ge 0$ true from $R\ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

Remark. Replacing h_a with w_a, we get:

16) In ΔABC the following relationship holds:

$$\left(\frac{3r}{R}\right)^2 \le \frac{w_a^2}{bc} + \frac{w_b^2}{ca} + \frac{w_c^2}{ab} \le \frac{9R}{8r}$$

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Solution. **Lemma. 17) In ΔABC the following relationship holds:**

$$\frac{w_a^2}{bc} + \frac{w_b^2}{ca} + \frac{w_c^2}{ab} = \frac{s^2(172Rr + 62r^2 - 9s^2) - r^2(384R^2 + 260Rr + 41r^2)}{(s^2 + r^2 + 2Rr)^2}$$

Proof. Using identity: $w_a = \frac{2bc}{b+c}\cos\frac{A}{2}$, we get:

$$\sum \frac{w_a^2}{bc} = \sum \frac{\left(\frac{2bc}{b+c}\cos\frac{A}{2}\right)^2}{bc} = \sum \frac{\frac{4b^2c^2}{(b+c)^2}\cos^2\frac{A}{2}}{bc} = 4\sum \frac{bc}{(b+c)^2}\frac{s(s-a)}{bc} =$$

$$= 4s\sum \frac{s-a}{(b+c)^2} = 4s\frac{[-9s^4 + s^2(172Rr + 62r^2) - r^2(384R^2 + 260Rr + 41r^2)]}{4s(s^2 + r^2 + 2Rr)^2} =$$

$$= \frac{s^2(172Rr + 62r^2 - 9s^2) - r^2(384R^2 + 260Rr + 41r^2)}{(s^2 + r^2 + 2Rr)^2}$$

Which follows from

$$\sum \frac{s-a}{(b+c)^2} = \frac{[-9s^4 + s^2(172Rr + 62r^2) - r^2(384R^2 + 260Rr + 41r^2)]}{4s(s^2 + r^2 + 2Rr)^2}$$

Let's get back to the main problem.

Using Lemma and $16Rr - 5r^2 \le s^2 \le 4R^2 + 4Rr + 3r^2 (Gerretsen)$, we get the conclusion.

Another way: Using $h_a \le w_a \le m_a$, we get:

$$\frac{h_a^2}{bc} + \frac{h_b^2}{ca} + \frac{h_c^2}{ab} \le \frac{w_a^2}{bc} + \frac{w_b^2}{ca} + \frac{w_c^2}{ab} \le \frac{m_a^2}{bc} + \frac{m_b^2}{ca} + \frac{m_c^2}{ab} \Leftrightarrow$$

$$\left(\frac{3r}{R}\right)^2 \le \frac{h_a^2}{bc} + \frac{h_b^2}{ca} + \frac{h_c^2}{ab} \le \left(1 + \frac{r}{R}\right)^2; \frac{9}{4} \le \frac{m_a^2}{bc} + \frac{m_b^2}{ca} + \frac{m_c^2}{ab} \le \frac{9R}{8r}$$

Equality holds if and only if triangle is equilateral.

18) In ΔABC the following relationship holds:

$$\frac{h_a^2}{bc}+\frac{h_b^2}{ca}+\frac{h_c^2}{ab}\le\frac{w_a^2}{bc}+\frac{w_b^2}{ca}+\frac{w_c^2}{ab}\le\frac{m_a^2}{bc}+\frac{m_b^2}{ca}+\frac{m_c^2}{ab}\le\frac{r_a^2}{bc}+\frac{r_b^2}{ca}+\frac{r_c^2}{ab}$$

Proposed by Marin Chirciu-Romania

Solution. **Using $h_a\le w_a\le m_a$ and $\frac{m_a^2}{bc}+\frac{m_b^2}{ca}+\frac{m_c^2}{ab}\le\frac{r_a^2}{bc}+\frac{r_b^2}{ca}+\frac{r_c^2}{ab}$, see inequality 15) we get the conclusion. Equality if and only if triangle is equilateral.**

19) In ΔABC the following relationship holds:

$$\left(\frac{3r}{R}\right)^2\le\sum\frac{h_a^2}{bc}\le\sum\frac{w_a^2}{bc}\le\sum\frac{m_a^2}{bc}\le\sum\frac{r_a^2}{bc}\le\frac{9}{4}\left(\frac{R}{2r}\right)^2$$

Proposed by Marin Chirciu-Romania

Solution. **See up these inequalities. Equality holds if and only if triangle is equilateral.**

Reference:

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