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ABOUT AN INEQUALITY BY OCTAVIAN PURCARU

Proposed by Marin Chirciu – Romania

1) Let be $x, y, z \in [0, \infty)$, with $x + y + z = 1$. Prove that:

$$3(xy + yz + zx) \leq 2(x^2 + y^2 + z^2)(x + y + z) + 9xyz$$

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Solution

Homogenizing the inequality that we have to prove, we write:

$$\begin{aligned} 3(xy + yz + zx)(x + y + z) &\leq 2(x^2 + y^2 + z^2)(x + y + z) + 9xyz \Leftrightarrow \\ \Leftrightarrow 2 \sum x^3 &\geq \sum yz(y + z), \text{ which follows from summing the inequalities having the} \\ \text{following form: } x^3 + y^2 &\geq xy(x + y) \Leftrightarrow (x + y)(x - y)^2 \geq 0, \text{ obviously, with equality} \\ &\text{for } x = y. \end{aligned}$$

We deduce the inequality from enunciation, holds with equality for $x = y = z = \frac{1}{3}$.

Remark.

We can develop the inequality:

2) Let be $x, y, z \in [0, \infty)$, with $x + y + z = 1$ and $n \geq 2$. Prove that:

$$(n + 1)(xy + yz + zx) \leq n(x^2 + y^2 + z^2)(x + y + z) + 9xyz$$

Proposed by Marin Chirciu – Romania

Solution

Homogenizing the inequality that we have to prove, we write:

$$\begin{aligned} (n + 1)(xy + yz + zx)(x + y + z) &\leq n(x^2 + y^2 + z^2)(x + y + z) + 9xyz \Leftrightarrow \\ \Leftrightarrow n \sum x^3 &\geq \sum xy(y + z) + 3(n - 2)xyz, \text{ which follows from summing the inequalities} \\ 2 \sum x^3 &\geq \sum yz(y + z) \text{ (1) and } (n - 2) \sum x^3 \geq 3(n - 2)xyz \text{ (2)} \end{aligned}$$

Let's prove the inequalities (1) and (2)

Inequality (1): $2 \sum x^3 \geq \sum yz(y + z)$, it follows from summing the inequalities having the form: $x^3 + y^2 \geq xy(x + y) \Leftrightarrow (x + y)(x - y)^2 \geq 0$, obvious, with equality for $x = y$. Inequality (2): $(n - 2) \sum x^3 \geq 3(n - 2)xyz$, it follows from the condition $n \geq 2$ and $\sum x^3 \geq 3xyz$, true from means inequality or from the identity:

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$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

We deduce the inequality from enunciation, holds with equality for $x = y = z = \frac{1}{3}$.

Reference:

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